

# ***A Guide to Effective Instruction in Mathematics***

## ***Kindergarten to Grade 6***

A Resource in Five Volumes  
from the Ministry of Education

### **Volume Two** **Problem Solving and Communication**

*Every effort has been made in this publication to identify mathematics resources and tools (e.g., manipulatives) in generic terms. In cases where a particular product is used by teachers in schools across Ontario, that product is identified by its trade name, in the interests of clarity. Reference to particular products in no way implies an endorsement of those products by the Ministry of Education.*

Une publication équivalente est disponible en français sous le titre suivant :  
*Guide d'enseignement efficace des mathématiques, de la maternelle à la 6<sup>e</sup> année.*

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# Introduction

This is Volume Two of the five-volume reference guide *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6*. This volume includes Chapters 5 and 6.

Chapter 5: Problem Solving focuses on the most effective method for developing and consolidating students' understanding of mathematical concepts in the primary and junior grades – that of teaching both *through* problem solving and *about* problem solving. Chapter 6: Communication emphasizes the importance of promoting oral communication about mathematics in the primary and junior grades, and describes a number of cross-curricular literacy strategies that foster “math talk” and, later, math writing in the classroom. (See the Introduction of Volume One for a summary of the organization and contents of the complete five-volume guide.)

A list of suggested professional resources for teachers and administrators is included in Volume One. It is meant to provide useful suggestions, but should not be considered comprehensive. A glossary of terms used throughout the guide is also provided at the end of Volume One. References are listed at the end of each individual volume.

This guide contains a wide variety of forms and blackline masters, often provided in appendices, that teachers can use in the classroom. Electronic versions of all of these materials can be found at [www.eworkshop.on.ca](http://www.eworkshop.on.ca). These electronic forms and blackline masters are in a Word format that can be modified by teachers to accommodate the needs of their students.

Go to [www.eworkshop.on.ca](http://www.eworkshop.on.ca) for electronic versions of the forms and blackline masters provided throughout this guide. They can be modified to meet classroom needs.

## Locating Information Specific to Kindergarten, Primary, and Junior Students in This Guide

An important feature of this guide is the inclusion of grade-related information and examples that help clarify the principles articulated. Such information is identified in the margins of this guide by means of icons referring to the relevant grades – K for Kindergarten, Grades 1–3 for primary, Grades 4–6 for junior. Examples and other materials that are appropriate for use at more than one level or are applicable to more than one level are identified by the appropriate combination of icons.

K – Kindergarten  
1–3 – Primary  
4–6 – Junior

Look for the following icons in the margins of this document:

**K**

**1-3**

**4-6**



# 5. Problem Solving

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# Problem Solving

## Introduction

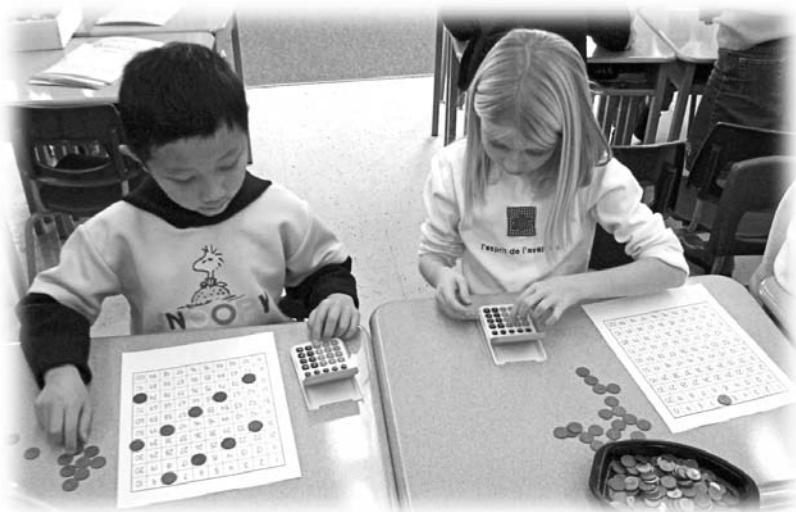
### THE IMPORTANCE OF PROBLEM SOLVING

An information- and technology-based society requires individuals who are able to think critically about complex issues, people who can “analyze and think logically about new situations, devise unspecified solution procedures, and communicate their solution clearly and convincingly to others” (Baroody, 1998, p. 2-1). To prepare students to function in such a society, teachers have a responsibility to promote in their classrooms the experience of problem-solving processes and the acquisition of problem-solving strategies, and to foster in students positive dispositions towards problem solving.

*“Problem solving is not only a goal of learning mathematics but also a major means of doing so.”*

(NCTM, 2000, p. 52)

In promoting problem solving, teachers encourage students to reason their way to a solution or to new learning. During the course of this problem solving, teachers further encourage students to make conjectures and justify solutions. The communication that occurs during and after the process of problem solving helps all students to see the problem from different perspectives and opens the door to a multitude of strategies for getting at a solution. By seeing how others solve a problem, students can begin to think about their own thinking (metacognition) and the thinking of others and can consciously adjust their own strategies to make them as efficient and accurate as possible.



In their everyday experiences, students are intuitively and naturally solving problems. They seek solutions to sharing toys with friends or building elaborate structures with construction materials. Teachers who use problem solving as the focus of their mathematics class help their students to develop and extend these intuitive strategies. Through relevant and meaningful experiences, students develop a repertoire of strategies and processes (e.g., steps for solving problems) that they can apply when solving problems. Students develop this repertoire over time, as they become more mature in their problem-solving skills. The problem-solving processes that Kindergarten students use will look very different from those that Grade 6 students use. Initially, students will rely on intuition. With exposure, experience, and shared learning, they will formalize an effective approach to solving problems by developing a repertoire of problem-solving strategies that they can use flexibly when faced with new problem-solving situations.

In fostering positive dispositions in their students towards problem solving, teachers deal with the affective factors that have an impact on student behaviour in both positive and negative ways (Schoenfeld, 1992). Students who believe that they are good problem solvers are not apt to give up after a few minutes when faced with a challenging problem. Because beliefs influence behaviour, effective mathematics programs always consider students' beliefs and attitudes, and teachers work to nurture in students confident attitudes about their abilities as mathematical problem solvers and their beliefs that everyone can make sense of and do mathematics.

As students engage in problem solving, they participate in a wide variety of cognitive experiences that help them to prepare for the many problem-solving situations they will encounter throughout their lives. They:

- learn mathematical concepts with understanding and practise skills in context;
- reason mathematically by exploring mathematical ideas, making conjectures, and justifying results;
- reflect on the nature of inquiry in the world of mathematics;
- reflect on and monitor their own thought processes;
- select appropriate tools (e.g., manipulatives, calculators, computers, communication technology) and computational strategies;
- make connections between mathematical concepts;
- connect the mathematics they learn at school with its application in their everyday lives;
- develop strategies that can be applied to new situations;
- represent mathematical ideas and model situations, using concrete materials, pictures, diagrams, graphs, tables, numbers, words, and symbols;

*“We want children to take risks, to tackle unfamiliar tasks, and to stick with them – in short, to try and persevere. We want children to be flexible in their thinking and to know that many problems can be modeled, represented, and solved in more than one way.”*

(Payne, 1990, p. 41)

- go from one representation to another, and recognize the connections between representations;
- persevere in tackling new challenges;
- formulate and test their own explanations;
- communicate their explanations and listen to the explanations of others;
- participate in open-ended experiences that have a clear goal but a variety of solution paths;
- collaborate with others to develop new strategies.

Problem solving is central to learning mathematics. In the words of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, "instruction based on a problem-solving or investigative approach is the means by which Ontario students will most readily achieve mathematical literacy" (2004, p. 9). Problem solving is an integral part of the mathematics curriculum in Ontario and is the main process for helping students achieve the expectations for mathematics outlined in the curriculum documents because it:

- is the primary focus and goal of mathematics in the real world;
- helps students become more confident mathematicians;
- allows students to use the knowledge they bring to school and helps them connect mathematics with situations outside the classroom;
- helps students develop mathematical knowledge and gives meaning to skills and concepts in all strands;
- allows students to reason, communicate ideas, make connections, and apply knowledge and skills;
- offers excellent opportunities for assessing students' understanding of concepts, ability to solve problems, ability to apply concepts and procedures, and ability to communicate ideas;
- promotes the collaborative sharing of ideas and strategies, and promotes talking about mathematics;
- helps students find enjoyment in mathematics;
- increases opportunities for the use of critical-thinking skills (estimating, evaluating, classifying, assuming, noting relationships, hypothesizing, offering opinions with reasons, and making judgements).

*"A problem-solving curriculum, however, requires a different role from the teacher. Rather than directing a lesson, the teacher needs to provide time for students to grapple with problems, search for strategies and solutions on their own, and learn to evaluate their own results. Although the teacher needs to be very much present, the primary focus in the class needs to be on the students' thinking processes."*

(Burns, 2000, p. 29)

Problem solving needs to permeate the mathematical program rather than be relegated to a once-a-week phenomenon – the "problem of the week". In this guide it is not considered to be one approach among many; rather, it is seen as the *main* strategy for teaching mathematics. Problem solving should be the mainstay of mathematical teaching and should be used daily.

Not all mathematics instruction, however, can take place in a problem-solving context. Certain conventions of mathematics must be explicitly taught to students. Such conventions should be introduced to students as needed, to assist them in using the symbolic language of mathematics. Examples of mathematical conventions include operation signs, terms such as *numerator* and *denominator*, the decimal point, the numerals themselves, the counting sequence, the order of the digits, and the *is less than* ( $<$ ) and *is more than* ( $>$ ) signs.

## INSTRUCTIONAL IMPLICATIONS OF TEACHING THROUGH AND ABOUT PROBLEM SOLVING

Every problem that teachers give to students can serve more than one purpose. One major purpose is to explore, develop, and apply a conceptual understanding of a mathematical concept (teaching *through* problem solving). A second major purpose is to guide students through the development of inquiry or problem-solving processes and strategies (teaching *about* problem solving).

When selecting instructional problems, teachers should ask themselves:

- What is the purpose of the lesson? Is it to develop a concept? Is it to develop problem-solving processes or strategies? Or is it to develop both?
- What problem would help children to learn the concept or to develop the processes or strategies?
- Will the context of the problem be meaningful to all students and engage them in a meaningful way?
- What questions or prompts will be used to support the development of the concept or of the processes or strategies?
- What kind of sharing and discussion about the concept or about the processes or strategies will happen at the end?
- What misconceptions might students have? How will misconceptions be addressed throughout and at the end?
- What form of assessment will provide information about student learning and the effectiveness of instructional practices?

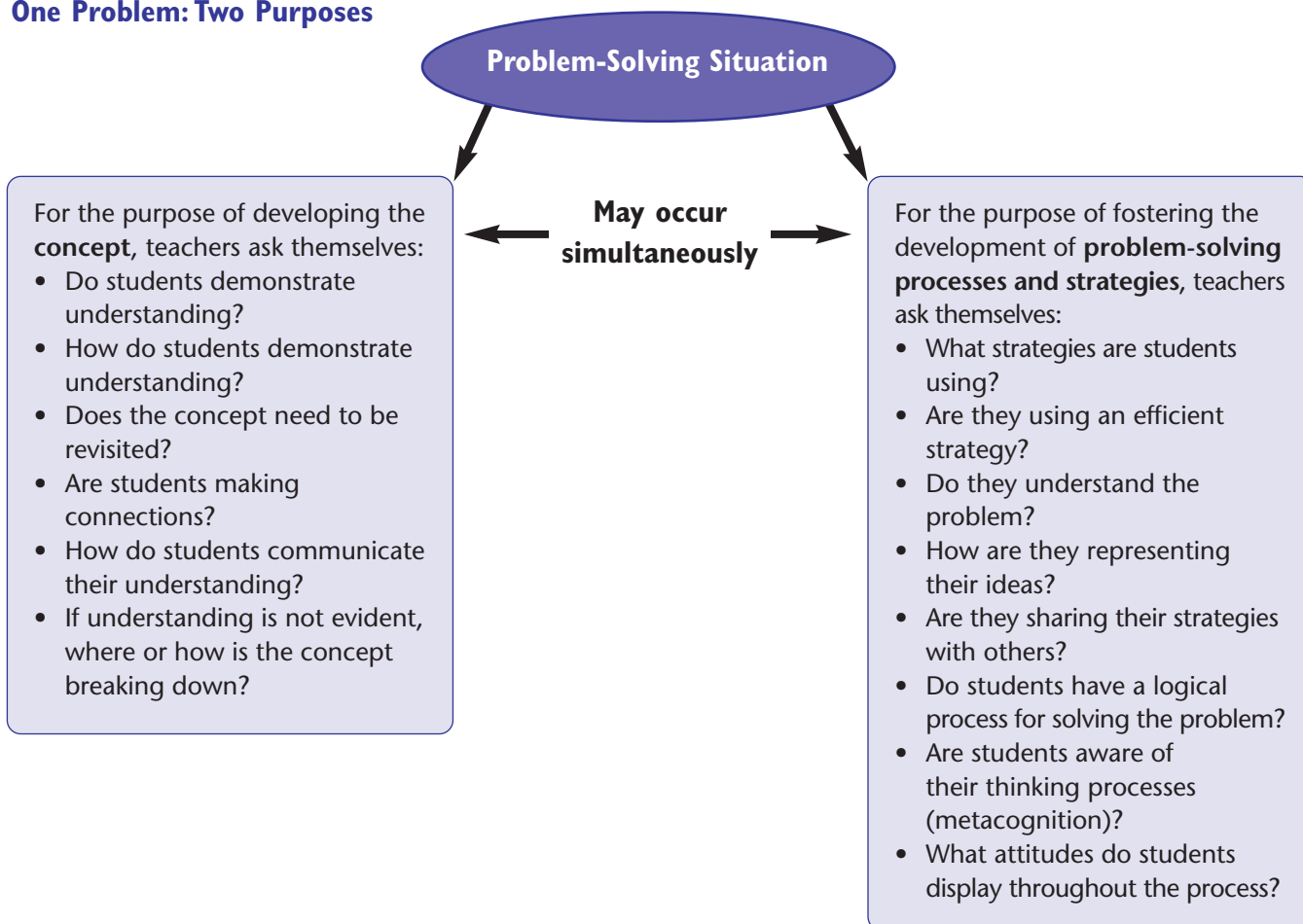
The chart on the opposite page outlines how one problem can be used both for the purpose of concept development and for the purpose of process or strategy development. While students are solving a problem, the teacher is observing for both purposes and using the questions in the chart to guide his or her observation. The teacher has to monitor for both purposes throughout the lesson. For example, the

*“Waiting for mastery of foundational skills such as arithmetic before teaching problem-solving skills places children with math difficulties at considerable disadvantage in terms of their mathematical skill development (Lyon et al, in press).”*

(Expert Panel on Literacy and Numeracy Instruction for Students With Special Education Needs, p. 74)

teacher needs to stop part-way through a lesson to ask both conceptual questions such as "What are the characteristics of the shapes you are finding?" or "Are all the shapes the same?" and problem-solving questions such as "What strategies are you using?" or "Is there a different way?"

### One Problem: Two Purposes



In this chapter, the subject of problem-solving instruction is addressed under the headings "Teaching Through Problem Solving", "Teaching About Problem Solving", and "Observing Students and Assessing Their Work as They Solve Problems". Examples of how problem solving can be used as a teaching strategy are included in the learning activities provided in the companion documents to this guide that focus on the individual strands.

## Teaching Through Problem Solving

Teaching through problem solving involves using problems as the medium for teaching mathematical content. Teachers pose problems at the beginning of a unit, throughout the unit, and at the end of the unit. They present engaging problems to students as a way of motivating students to investigate mathematical concepts and to develop and apply their own understanding of those concepts. Most concepts and computations can be taught through problem solving.

The following is an example of a problem that could be used to help Grade 2 students explore the addition of two-digit numbers. The instruction planned for this problem asks students to utilize their own strategies and knowledge in order to develop new understandings and create connections. This is the problem:

How many pieces of pizza do we need to order for our class and Mrs. Smith's class?

This problem sets up a scenario in which students are motivated to count the number of students in one class, count the number of students in another class, and combine the totals. Students could use a variety of methods and manipulatives to solve the problem (e.g., counting on with a hundreds chart, modelling with base ten materials, or writing down with paper and pencil). After students have arrived at solutions, the class would reassemble to share, and the teacher would record methods and guide the discussion. Various students would share the strategies they used to complete the calculation.

In Grade 6, students might be asked to solve the following problem to help them investigate concepts about ratio:

The manager of a pizza store recommends that you order 2 pizzas for every 5 students. If you followed this recommendation, how many pizzas would you need to order for a party for 38 students?

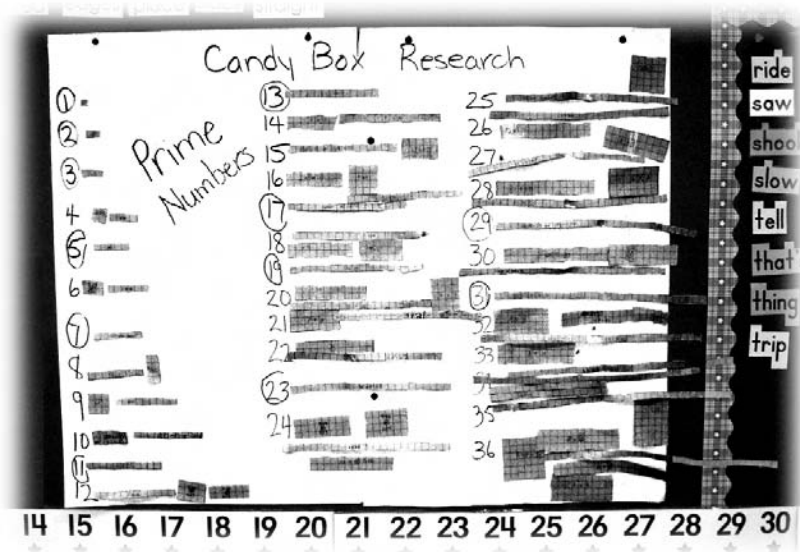
Before students have been taught formal procedures for solving problems about ratio, they use intuitive strategies that make sense to them. Some students might, for example, use concrete materials or drawings to represent pizzas and students. Others might construct a table similar to the one shown below to show the relationship between pizzas and students.

Pizzas	Students
2	5
4	10
6	15
8	20
10	25
12	30

As students solve the problem, they learn that a ratio (a relationship between two quantities, as in 2 pizzas per 5 students) can be used to produce equivalent ratios. The problem-solving experience helps students develop the ability to reason about ratios in meaningful ways, before they are introduced to symbolic and mechanical methods of solving problems about ratio.



In teaching through problem solving, as evidenced in the examples above, the teacher moves away from a “watch me” method of teaching towards a method in which students are actively involved and challenged, and in which they use representations (concrete and graphic models, pictures, or diagrams) to gain a deeper understanding of mathematics.



## WHAT TEACHING THROUGH PROBLEM SOLVING LOOKS LIKE

1-3

4-6

The following vignettes help to paint the picture of what teaching through problem solving looks like. In the first vignette, Grade 1 students explore how a number can be decomposed (broken down into parts). In the second vignette, Grade 4 students explore the concepts and relationships of perimeter and area. The lessons are structured in three phases – “Getting Started”, “Working on It”, and “Reflecting and Connecting”. The left-hand side of the page is used for telling the story of what the teacher and students say and do when a problem is used for instructional purposes. The right-hand side of the page is used for describing the actions and decisions of the teacher throughout the problem-solving process. Not included or described in these vignettes are pre-task decisions (e.g., how the tasks connect with long-range plans and unit plans). The classroom description provides examples of types of on-the-spot decisions that teachers make in response to student ideas that occur throughout the problem-solving process.

# Getting Started (Preparing for learning)

This shared experience allows the teacher to engage all students as they explore a new mathematical concept in a familiar context.

The teacher presents the problem. Initial instructions for completion of the task are clearly outlined. Before students are sent off to work, the teacher checks to ensure that they understand the problem.

The problem used is more than just a word problem; it is a problematic situation that allows students to explore the concept of decomposing: in this case, the number 8 (e.g., 8 can be decomposed as 8 and 0, 7 and 1, or 6 and 2, and so on).

*"Today, I want your help in solving a problem.*

*"I just received 3 more gerbils and now have a total of 8 small brown gerbils. I know that all 8 gerbils cannot live in the same cage – it would be too crowded. So I bought another cage over the weekend."*

The teacher shows the students two equal-sized sheets of paper and explains that the sheets of paper represent the two cages.

*"Before I make a decision about how many gerbils to move to the new cage, I want to know all the different number combinations for the two cages. Right now, I have 8 and 0 – 8 gerbils in one cage and 0 gerbils in the other cage. But I know that one cage is too crowded. What are other ways I can put my 8 gerbils in two cages?"*

*"You can use any material that makes sense to you in solving this problem. You can work with a partner or on your own today.*

*"So let's consider the question again."*

The teacher shows the problem posted on chart paper. *"Who will tell us what the problem is?"*

This is a quality problem because it:

- addresses a cluster of curriculum expectations;
- builds upon the "big idea" of *quantity* by asking students to decompose the number 8;
- helps students to recognize numbers as different combinations of other, smaller numbers (e.g., 10 is 0 and 10, 1 and 9, 2 and 8, 3 and 7, 4 and 6, or 5 and 5), an ability that will allow students to be more flexible when working with operations;
- involves the meaningful use of mathematics;
- allows learners of varying abilities to enter into solving the problem (through the use of manipulatives, the open-ended nature of the problem, and the potential for students to use their own experiences to solve the problem);
- is rich enough to allow for adaptations, extensions, and connections;
- is developmentally appropriate; that is, it is a reasonable problem for Grade 1 students to solve;
- is active – the student is able to decide how to approach the problem and to decide what materials to use;
- is collaborative – the student can interact with other students or the teacher in order to solve the problem.

By having one or more students describe the problem in their own words, the teacher is helping to ensure that all students understand the task at hand.



The teacher selects one or two students to share what they think the problem is asking. One student responds by saying, *"You have 8 gerbils and, um, two cages. I think we have to figure out how to put them into, um, . . . two cages."*

The teacher reminds students about the norms that they have developed for problem solving. *"Now remember, when you are working on your solution to the gerbil problem, you may use any of the math tools we have in the classroom. Some of you may find interlocking cubes, pattern blocks, or counters helpful for representing the problem. You may choose to work with a partner or by yourself. Remember, though, that sometimes working with a partner can be helpful, as you may learn a new way to solve the problem."*

One student asks, *"Will we have to show our work to the rest of the class?"* "Yes," responds the teacher, *"you and your partner need to be ready to share your representation with the class. That might be as a picture or it might be with the manipulatives you've used."*

"Also," reminds the teacher, *"remember that if you become stuck and aren't sure what to do, it's a good idea to ask someone else for an idea before asking me. Some of your classmates will find exciting ways to solve the problem, and I'm sure they will be willing to help you with an idea or two to get you started."*

*"I'll be coming around to watch and listen as you work."*

*"In fact, I might even take some notes about the strategies you've used and the different ways you are able to put the gerbils in the two cages."*

Some students return to their tables, while other students work on the floor around the room and try to solve the problem.

By reminding students of the class problem-solving routines, the teacher encourages students to work autonomously and to utilize the knowledge of other students to help them become unstuck.

The teacher listens carefully to be sure that students recognize that the problem is intended to tap into their understanding of decomposing 8 (e.g., into 7 and 1, 6 and 2).

The teacher tells students what he or she is looking for when observing them. The students' strategies and solutions are also the focus of what the teacher records in the anecdotal notes.

## Working on It (Facilitating learning)

The teacher facilitates learning by:

- providing situations in which students are trying their own strategies;
- offering guidance and redirection through questioning;
- giving assistance to those who require it and allowing the others to solve the problem independently.

In this *independent* and *shared* experience, the teacher encourages students to choose an appropriate manipulative that will assist them in representing the situation. The teacher reminds students of some of the manipulatives that may be useful (interlocking cubes, pattern blocks, or counters) but does not suggest that they are the only manipulatives available. By encouraging students to make their own choices, the teacher affords all students the opportunity to choose and test their own strategies, and to construct their own understanding of the problem and its solutions.

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At this point the teacher is circulating around the room and doing a visual check to make sure that students are on task. Some students require clarification, others assistance to gather materials, others a place to work.

### *Working in Pairs/Groups*

Although some students choose to work with a partner, each individual student is responsible for recording his or her own thinking and solutions. The teacher can use these recorded work samples when questioning students individually to ensure conceptual understanding. By using a journal or log as a recording device throughout the year, teachers can refer to anecdotal notes that show growth in problem solving, communication, and understanding. Such a journal or log could be separated into sections, so that each student has a section in the journal where specific anecdotal comments and observations can be recorded.

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After a few minutes the teacher claps a pattern and the students stop and listen. *"I want you to find another student or group and talk about how you started to solve this problem. What were you thinking, and what strategy do you think you are going to use?"*

Students find others with whom to share their initial thinking and then return to their workspaces to continue working on the problem. The teacher hears two students who have decided to modify their strategy to include using interlocking cubes on the basis of what they heard from their friends during the show-and-tell time.

Having students share their initial thinking and hear other students' strategies early on and throughout the problem-solving process helps to validate strategies and trigger new ideas, and provides hints for those students who are stuck.

The teacher comments on the selection of materials and some of the strategies being used.

*"I'm noticing that you are using many different manipulatives. That's great!"*

The teacher moves on and finds that Hallie has already solved the problem. Hallie has arranged tiles to show all the possible number combinations and has orally explained her reasoning.

### *Representing With Manipulatives, Pictures, or Numbers*

For this particular task, the teacher has given students the opportunity to select the manipulative that will help them represent and make sense of the problem and solve it. Students who select marbles to model the gerbils in the problem discover for themselves that a different manipulative will better help them to illustrate the problem and keep track of their number combinations. Some students may approach this problem by using pictures and/or numbers to represent their combinations, and not use manipulatives at all. In other words, for some students using a manipulative may not be the only way to model the problem and a solution. Such students may find that a pictorial or numbered representation is an effective way to model the problem.

The teacher gets the attention of the class. *"I would like to remind you to make a record of your work. On this paper, show how you solved the problem. If you're working with a partner, you will each need your own record of your work. Remember all the different ways we have found to show our thinking."*

When students are working in groups, it is important that the teacher gather information about each individual student's ideas, understandings, and processes.

The teacher goes back to Hallie and offers her a modified problem. *"Hallie, your explanation shows me that you understand the problem. What if I had three cages and 8 gerbils? What are the different ways that I could now arrange my gerbils?"*

After a pause the teacher asks, *"What do you think you'll find?"*

*"I'm going to need another cage,"* she responds.

*"You're right!"*

Hallie uses another piece of paper to represent the additional cage and begins to rearrange her tiles to solve this new problem.

The teacher continues to circulate through the room and notices that two students, Anju and Michael, have not yet begun solving the problem. *"Why don't the two of you come over to the carpet?"*

### *Adaptations to the Lesson*

The teacher adapts the task to address the various needs of students.

The first adaptation (for Hallie) involves an extension of the task. The task is adapted to challenge the student who quickly solves and understands the initial problem.

The second adaptation (for Anju and Michael) requires that the teacher scaffold the task through leading questions. The teacher works together with the two students to model the first two solutions, and then the students are left to complete the task. Changing the number of gerbils and providing intensive guidance would be an appropriate adaptation of the task for students who are not meeting the curriculum expectations at their grade level.

*What do we know about this problem? Let's write this down on the chart paper. What manipulatives are we going to use to represent our gerbils? What else do we know? What could we use for the cages? What is one way of placing our gerbils in the cages? Let's record this on the chart paper. Good. Is there another way? Let's record it so we don't forget. I'll write your ideas to begin with. Work together to find more ways to arrange the gerbils. I'll come back in a few minutes to see how you're doing."*

On an anecdotal tracker, the teacher is recording notes about the number of combinations that students have found as well as the way that some students are beginning to record the information in an organized way.

The teacher continues to walk around and observe, and begins to record some of the assessment information that has been gathered to this point.

*"You have ten more minutes to work on the problem before we share what we've found."*

When planning for this task, the teacher considers the different needs and learning styles of students and considers appropriate adaptations or designs alternative experiences before presenting the problem, so that they are at hand when required.

To provide differentiated instruction, the teacher poses guiding questions to individuals as they work through the problem.

### **Reflecting and Connecting** (*Reflecting on, extending, and consolidating learning*)

In this very important part of the experience, the teacher leads a discussion in which students share their strategies, consider their solutions to the problem, and determine which make most sense. Enough time is allotted to allow for the sharing of several examples. This discussion validates the various strategies used and consolidates learning for students. For many students, the discussion, questioning, and sharing that occur during this phase allow them to make connections with their own thinking and to internalize a deeper understanding of the mathematical concepts. However, the completion of a rich task does not ensure that students' understanding has been developed, since misconceptions can still be evident in students' ideas.

The teacher assembles students as a group on the carpet in such a manner that all students have the opportunity to see and hear one another.

*"Who would like to tell us what the problem was that we just solved?"*

One student restates the problem in his or her own words.

Two students who worked together move to the front of the group and hold up their recordings to share with the class.

Sergio chooses one of his number combinations to share. His partner chooses a different number combination from his own recording.

#### *Teacher Questioning*

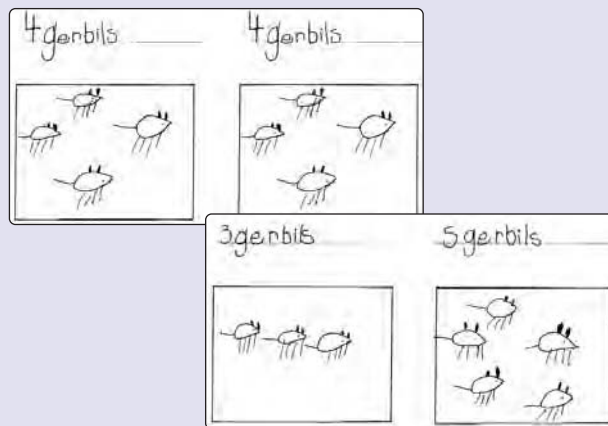
Here the teacher is asking questions to guide the discussion, to emphasize the mathematics, and to build clarity and understanding for all students. In particular, the teacher leads students in realizing the need for and usefulness of an organized list.

As students respond to the questioning, the teacher has the opportunity to find out what students know, where they will need further instruction, and what experiences to plan for next.

*"What strategy did you use to find your solutions?" the teachers asks. "Did anyone find a different way of arranging the gerbils?"*

Students continue sharing until a student points out that a particular number combination has already been shared.

*"Are we sure that we've seen this number combination before? We need to keep track. How can we do that? How do we know we have them all?"*



Some students suggest writing down the number combinations or posting the combinations that have been shared already so that they can be referred to.

*"Patty and Alima recorded their findings by making a list. Why don't we use that strategy to get us started?"*

### *Problem-Solving Strategies*

Most students will use the strategies of acting it out or using a model to solve the gerbil problem. The teacher highlights the strategies used when students share their solutions and models the strategy of making an organized list to keep track of all the possible combinations. Since making an organized list can be a challenging strategy, the teacher will need to model it, thinking aloud to help students begin to assimilate it.

The teacher leads students in finding those number combinations that have already been shared and adding different ones to the chart. This process continues until a student notices what he or she considers to be a repeat. *"We have that one up there two times, 'cause 6 and 2 and 2 and 6, that's the same thing!"*

*"What do the others think? Are 2 and 6 and 6 and 2 the same thing in this problem?"*

Realizing that this is a difficult concept for some students, the teacher asks a student to show his or her representation of 2 and 6. The teacher then physically turns the sheet of paper around 180 degrees so that the students can see that 2 and 6 and 6 and 2 really are the same representation.

After listening to a variety of viewpoints, the teacher uses demonstration-sized cubes to model the combinations.

Demonstration-sized manipulatives help to ensure that all students can see the representation. Overhead and, in some cases, magnetic manipulatives are also available.

*"Are there any more repeated number combinations? Let's check the chart."* After students have shared all the combinations of numbers they have come up with, the teacher says: *"Let's see if we've found all the different ways of arranging the numbers."*

### *Assessing Students' Learning*

During this discussion and while students are working to solve the problem, the teacher is looking and listening for:

*Let's organize what we've found to make sure we haven't missed anything. Let's start with 0 and 8. What combination would come next? 1 and 7 makes sense.*

*"Making a list is a great way to help organize the combinations. I think we should add this strategy to our strategy wall. I'm sure we'll run across other problems this year where you might want to try making a list to help organize your ideas."*

After students have agreed that they have all the number combinations, the teacher suggests that they test their ideas. The teacher makes two large rectangles (two sheets by two sheets each) of newspaper on the floor and asks 8 students to stand in one of the large rectangles. Students then act out the problem in order to prove that they have all the possible combinations.

Once it has been decided that all the possible number combinations have been found, the teacher takes students back to the original problem and asks them to decide which combination would be the most suitable. Many students agree that the 4 and 4 split would be appropriate.

The teacher also provides an opportunity for students to reflect on what they have learned.

*"We have learned lots about the number 8 today by solving the gerbil problem. Talk with the person beside you about everything you've learned about the number 8."* After a few minutes, a few students share their ideas with the group.

*"Let's look at our number combinations. What do you think would happen if we had 9 gerbils instead of 8? I think that is a great question for you to explore at home. It will be your job to explain carefully to your parents the gerbil problem from today and to work on a solution to the problem of how many ways we can put 9 gerbils in two cages. Be sure to talk to your parents about all the strategies you used to solve the problem in class."*

- evidence of the problem-solving strategies that students are using;
- representations and justification of solutions.

On other occasions, the teacher might be making an assessment based on other criteria, such as the accuracy of solutions, the organization of the information, the explanation of procedures, the demonstration of understanding, or the use of mathematical language.

Since this problem introduces a new concept, the assessment information collected is used to plan next steps and future learning experiences. When a problem is used as a culminating task, the information gathered could be used to evaluate students and report to parents.

#### *Home Connection*

Sending the students home with a "job" serves several purposes and offers various benefits:

- Parents/caregivers are informed.
- Parents/caregivers are involved in their child's math learning.
- Students have the experience of explaining the process and strategies they used to solve the problem.
- Students feel as though they are doing math for a purpose.

The teacher provides for every student a “Home Connection” page that explains to parents how the class has been investigating the number 8 by decomposing it into all the number combinations possible. The sheet describes the problem with 9 gerbils and asks parents to work on a solution with their child.

When a “Home Connection” is sent, it is important that background information for parents be included, so that they see how the question or task connects with the mathematical concepts being investigated at school. When the home task is kept within a familiar context, students are likely to be able to work confidently on the task.

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*“Tomorrow we’ll have a chance to share what you discovered and discuss it further.”*



### Getting Started (Preparing for learning)

In this lesson, a Grade 4 teacher presents a problem that will provide students with an opportunity to explore the concepts and relationships of perimeter and area. To spark the students' curiosity and to prepare them for the problem-solving experience, the teacher explains, *"When you come in from recess, please come over to our group area. I have a message that I would like to share with you."*

After recess, the teacher will present the problem and lead a discussion with the students.

Stephanie, remembering what the teacher told the class before recess, asks, *"What's the message?"*

The teacher explains: *"The principal left a message that our class will be receiving a new carpet for the classroom. The principal has given us a job to do – she has asked us to clear a space so that the carpet can be delivered tonight. The principal also said that the perimeter of the carpet is 12 m."*

*"That's big,"* says Joel. *"We can't fit a 12 m carpet in this room!"*

Damon adds, *"No, it's not 12 m long. It has a perimeter of 12 m. That's the distance around the carpet."*

Joel asks, *"How many metres long is it?"*

The teacher replies, *"That's a good question. Unfortunately, I don't know the answer, and the principal is away today."*

This is a quality problem because it:

- addresses a cluster of curriculum expectations;
- integrates concepts from different strands – Number Sense and Numeration, Measurement, and Geometry and Spatial Sense;
- helps students recognize the relationships between perimeter and area;
- involves the meaningful use of mathematics;
- allows learners of varying abilities to enter into solving the problem (through the use of manipulatives, the open-ended nature of the problem, and the potential for students to use their own experiences to solve the problem);
- is rich enough to allow for adaptations, extensions, and connections;
- is developmentally appropriate; that is, it is a reasonable problem for Grade 4 students to solve;
- is active – the student is able to decide how to approach the problem and to decide what materials to use;
- is collaborative – the student can interact with other students or the teacher in order to solve the problem.

*"Can someone tell us again, what we do know and what we need to try to figure out?"*

One student explains: *"We know we have a new carpet coming to our classroom tonight, but the problem is we don't know how big it is. We don't know how much space it will take up in the classroom."*

*"You're right, Lena. That is the problem. Can someone add some more information to what we have? We know that the carpet has a perimeter of 12 m, and Damon reminded us that perimeter is the distance around."*

By having students restate the problem in their own words, the teacher is helping to ensure that all students understand the task at hand.

Discussion about the problem helps students activate prior knowledge and gives the teacher valuable information about what students already understand.



Several students respond:

- *"We don't know the shape, but most carpets are rectangles."*
- *"It could be a square."*

The teacher asks, *"Do we have enough information to solve this problem?"*

A student answers, *"If we had more information, it wouldn't be a problem!"*

On the board, the teacher records the information that the students know (the carpet has a perimeter of 12 m), and the information that they want to find out (the size of the carpet). The teacher asks the students to find as many solutions as possible, so that the class can decide which solution or solutions are the most likely.

The teacher directs students to work with a partner or individually. The teacher reminds the students, however, that each one is responsible for recording and sharing what he or she discovers.

Although some students choose to work with a partner, each individual student is responsible for recording his or her own thinking and solutions. By asking the students to record their own work, the teacher encourages all students to think about strategies and solutions for the problem. The students' recorded work also provides the teacher with assessment information about each student's understanding.

Before going on, the teacher asks, *"Does everyone understand the problem well enough to get started?"* Everyone seems to understand.

*"Remember the next phases in solving a problem. When you think you understand the problem, you need to come up with a plan and then try it out."*

The teacher refers to a chart that outlines the four phases in the problem-solving process:

- understanding the problem
- making a plan
- carrying out the plan
- looking back and reflecting on the solution

The teacher adds, *"Remember that you may use any of the materials in the classroom to help you make and carry out your plan. Do any of you know already which materials you will use?"*

Students respond:

- *"I want to use the pattern blocks. I remember using them to make patterns in a carpet last year, so maybe they can help me on this problem."*
- *"I am going to use the square tiles. They can help us to make rectangles and squares."*
- *"Can I use a big sheet of the grid paper?"*

#### *Representing With Manipulatives or Concrete Materials*

Students who have had many opportunities to work with a variety of manipulative materials build on past experiences and extend the learning to new situations.

Manipulative materials should be readily available to students. Teachers should demonstrate the use of manipulative materials often to show students that they are valuable tools in doing mathematics.

For this particular task, the teacher gives students the opportunity to select a manipulative that helps them represent and solve the problem.

The teacher asks, *“What can you do if you get stuck as you’re working on the problem?”*

The students, remembering strategies from previous problem-solving experiences, explain that they could talk to a classmate or refer to the chart outlining the phases in the problem-solving process.

The teacher establishes problem-solving routines and encourages students to support one another during problem-solving experiences.

The teacher goes on to say, *“I will be coming around the room as you work. I may ask some questions and take some notes. We will stop after a short time to share some of your plans.”*

The teacher reminds students that they need to be prepared to talk about their work with the teacher and with other students.

### **Working on It** (*Facilitating learning*)

The teacher facilitates learning by:

- providing situations in which students are trying their own strategies;
- offering guidance and redirection through questioning;
- giving assistance to those who require it and allowing the others to solve the problem independently.

In this *independent* and *shared* experience, students develop and carry out strategies to solve the problem. A variety of manipulative materials are available, and the teacher encourages students to select the materials that will help them work towards a solution. By encouraging students to make their own choices, the teacher affords all students the opportunity to choose and test their own strategies, and to construct their own understanding of the problem and its solutions.

After about ten minutes, the teacher asks the students to stop working. *“Thanks for stopping so quickly. I know you are not finished, but before we go any further, let’s share how we got started. I noticed as I was walking around that you are using different strategies. Please share what you are doing with a neighbour. Then we will share some ideas together.”*

The teacher provides an opportunity for students to talk about their strategies with a classmate before they share their ideas with the large group.

After a few moments of sharing, the teacher calls for the students’ attention. *“Let’s share our ideas. It’s important that we listen attentively to one another, so that we can learn from one another. Who can share what he or she is doing?”*

The teacher waits until all students have stopped talking and are giving their full attention. Listening attentively, a skill that develops over time, is an important aspect of an effective learning community.

Some students explain that they are using geoboards; others say that they are drawing pictures; still others are arranging square tiles.

*“Have you seen anyone use another effective strategy?”*

A student says, *“I saw that Travis and Kristen are using grid paper.”*

The teacher recognizes that students in the junior grades are able to identify and explain their problem-solving strategies. The teacher listens to their explanations to determine whether there are misconceptions, and notes any ideas that should be revisited later.

*"Travis and Kristen, can you explain your strategy, please?"*

The two students describe how they draw rectangles on the grid paper and try to work it out so that the perimeter is 12 units. *"Some of the rectangles don't work,"* admits Kristen, *"so we just put an X on the picture."*

*"What problem-solving strategy are Kristen and Travis using?"* The students agree that they are using a guess-and-check strategy.

Other students explain their strategies:

*"We know that the perimeter is 12 m, so we're subtracting the length of the sides from 12. Then we subtract the top and bottom. Sometimes it doesn't come out even, so I guess we're using a bit of guess and check as well."*

After several students have shared their progress on the problem, the teacher thanks those who have shared and invites the students to continue their work.

To demonstrate that all strategies are valued, the teacher expresses appreciation to the students who share their ideas.

The teacher approaches Kristen and Travis and discovers that they have successfully explored the possible solutions as well as several other inaccurate configurations of square tiles, using the guess-and-check method.

The teacher adapts the task to address the various needs of students.

The teacher offers an additional challenge. *"What if the carpet for the classroom had a perimeter of 14 m? How many possible arrangements of square tiles would create a carpet with this perimeter?"*

The first adaptation (for Kristen and Travis) involves an extension of the task. The task is adapted to challenge the student who quickly solves and understands the initial problem.

The teacher joins a student working on his own. *"Joel, I noticed that you are having some difficulties getting started today. Did any of the ideas that were just shared help you?"*

*Teacher Think-Aloud*

When Joel shrugs his shoulders, the teacher asks him to join Suzanne, who is working independently but is progressing slowly.

The second adaptation (for Joel and Suzanne) requires that the teacher scaffold the task by using leading questions and the think-aloud technique. The teacher works together with the two students to model one of the solutions suggested by their classmates, and then the students are left to complete the task. The teacher revisits this working group throughout the work period, ensuring that progress is being made and providing additional prompts and/or leading questions to further their progress.

The teacher works with the pair of students to review the important information provided in the problem and a few of the discoveries already made by their peers.

*"Let's write information that we know about the problem on the chart paper. What are you asked to do in the problem?"*

*"Find as many solutions for a carpet with a perimeter of 12 m as possible," Joel responds.*

The teacher prompts the students by asking them to recall one solution discussed with the group and to record it on the chart paper.

The teacher then models how to use a guess-and-test method, using a think-aloud technique to explore other possible dimensions for the carpet. *"If I could build a carpet with a perimeter of 12 m using 5 square tiles lined up in a row, can I use 6 square tiles in 2 rows?"*

The teacher demonstrates how the students might use square tiles but is careful not to give them a solution to the problem. The teacher allows them time to explore this possibility and then returns later to continue guiding the students, using the think-aloud technique if necessary.

After the students have worked for another fifteen minutes, the teacher lets them know that they have five minutes left to work on the problem. She reminds them to leave their work in such a way that others can see it.

Five minutes later, the teacher stops all work, saying, *"I would like you to join me at the group area, but on your way, please take a walk around the room and observe what other students have done. Some of the solutions are difficult to bring to our group area, but I want you to see all the exciting math that has happened here today!"*

When planning for the task, the teacher considers the different needs and learning styles of students and considers appropriate adaptations and/or designs alternative experiences before presenting the problem, so that they are at hand when required.

To provide differentiated instruction, the teacher poses guiding questions to individuals as they work through the problem.

The teacher is mindful of acknowledging and celebrating the mathematics that students are doing.

### **Reflecting and Connecting** *(Reflecting on, extending, and consolidating learning)*

In this important part of the lesson, the teacher leads a discussion in which students share their strategies, consider their solutions to the problem, and determine which solutions make sense. Enough time is allotted for the sharing of several examples. This discussion validates the various strategies used and consolidates learning for students. For many students, the discussion, questioning, and sharing that occur during this phase of the lesson allow them to make connections with their own thinking and to internalize mathematical concepts. However, the completion of the task does not ensure that students' understanding has been fully developed, since misconceptions can still be evident in students' ideas. Concepts need to be revisited at a later time.

The teacher asks for volunteers to share their work.

Two students show their drawing of a  $2 \times 4$  rectangle on grid paper. *"We had more than one solution, but we think the carpet will be a rectangle that looks like this."*

The teacher reminds them that they need to tell *why* they think this is the solution and to explain *how* they arrived at this answer.

One of the students continues: *"We have seen lots of carpets that are this shape, so that part just makes sense. We have the numbers 1 to 12 around the outside. Each number represents 1 m, so we have a carpet that has a perimeter of 12 m. The carpet that we drew is 2 m at the top and bottom and 4 m on the sides."*

The teacher asks, *"Is there anyone else who thought the carpet could be this shape and size?"*

Hands go up. A pair of students show a  $2 \times 4$  rectangle on their geoboard.

Another pair show a diagram of a  $2 \times 4$  rectangle, as well. The students explain that they used square tiles to help them make the rectangle.

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The teacher guides the discussion by asking probing questions. *"We had more than one group show the same shape and size, but what did you notice?"*

Students respond that the groups used different materials (e.g., geoboards, graph paper, square tiles, interlocking cubes) and strategies (e.g., guess and check, make a model, draw a diagram, use an organized list) to solve the problem.

### *Clear Expectations*

Students are reminded that they are expected not only to share their solution but also to explain the strategies and processes they used to arrive at this solution. Given appropriate opportunities, junior students will develop the ability to explain their mathematical thinking and justify their solutions.

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### *Problem-Solving Strategies*

It is important that the teacher encourage and accept a variety of problem-solving strategies. Students should be able to explain why they chose a particular strategy and to evaluate how effective the strategy was in helping them find a solution to the problem.

Most students will use a combination of make a model, guess and check, and draw a diagram to solve the perimeter problem. Some students may choose to use an organized list to chart their possible and impossible solutions. Since making an organized list may be challenging for some students in the junior grades, the teacher may wish to use this opportunity to model this strategy if it is not demonstrated in any of the students' solutions.

The teacher invites other students to share their findings and explain the strategies they used. *"Did anyone find that the carpet could have a different area or shape but have the same perimeter?"*

A student hesitantly responds, *"Our carpet looks different, but the numbers are kind of the same. We thought the carpet would be wide, not long."*

A student explains, *"That's the same as our carpet, but it's turned a different way."*

A different group offers their suggestion: *"We think it will be a square carpet, so we divided 12 into four equal parts, because a square has four equal sides. So each side is 3 m long."*

The teacher uses this opportunity to guide students in thinking about the area of the carpets. *"We know our carpet will cover a part of the floor. How many square metres will the carpet cover? Let's use these square tiles to find the area of the carpet that is 3 m by 3 m."*

A student arranges the tiles in a  $3 \times 3$  array while the other students watch. The students recognize that the area of the square carpet is  $9 \text{ m}^2$ .

The teacher continues the discussion. *"Do all rectangular carpets with a perimeter of 12 m have an area of  $9 \text{ m}^2$ ? Think about the  $2 \times 4$  carpet we looked at earlier. Let's use the square tiles to find the area of that carpet."*

A student arranges the tiles in a  $2 \times 4$  array, and the other students soon find that the area is 8 square tiles.

The teacher prompts students to make further connections. *"In what way are the  $3 \times 3$  carpet and the  $2 \times 4$  carpet the same? How are they different?"*

*"They have the same number around but not the same number inside,"* answers one student.

Still probing, the teacher asks whether someone can explain the comparison in a different way.

*"They have the same perimeter but not the same area."*

#### *Teacher Questioning*

The teacher asks questions to guide the discussion, to emphasize mathematical concepts, and to help develop students' understanding. In particular, the teacher is helping students understand the measurement concepts of perimeter and area and the relationship between them. The teacher is also encouraging students to communicate their understanding effectively.

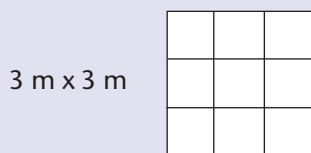
As students respond to the questioning, the teacher has the opportunity to find out what students know, where they will need further instruction, and what experiences to plan for next.

The teacher gives evidence of valuing the opinions and ideas of the students, and gives students opportunities to share with one another in a supportive environment. The teacher encourages students to use mathematical language and thinking.

The teacher redirects the class's attention. *"We need to make a class decision soon about the area to clear for our carpet. Who can tell us the different sizes and shapes of the carpets that we have talked about?"*

A student responds, *"We had one square and some rectangles but I'm not sure how many."*

The class decides to make a list that shows the dimensions of possible carpets. The teacher begins the list by recording  $3\text{ m} \times 3\text{ m}$  and drawing a small diagram to illustrate it.



Students suggest that the  $2 \times 4$  and  $4 \times 2$  rectangular carpets should be added to the list. The teacher includes these and draws diagrams for them as well.

Another student adds, *"We had another one. We have a rectangular carpet but ours is long and narrow with only 1 m at the top."* Another student comments, *"That would be really long!"* The teacher invites the students to draw and label a  $1 \times 5$  rectangle.

The students decide that they have recorded all possible carpet sizes in the list. The class agrees that although each solution could be a possibility, the most likely shape would be the rectangle that measures 2 m by 4 m. Next, the students use metre sticks to measure a 2 m by 4 m space in the classroom, and they realize that the carpet would fill the space in the classroom used for floor activities and sharing. Joel suggests, *"We should move some desks, as well, just in case the carpet is the 3 m by 3 m square."*

At the conclusion of the discussion, the teacher explains a task that the students are to complete at home. *"What if a rectangular carpet has a perimeter of 16 m? What are the possible dimensions for the carpet? Draw and label the length and width of each carpet on a piece of paper."* The teacher continues, *"Be sure to tell your parents all the things you learned about perimeter and area today as well as the problem-solving model and strategies that you used."*

*"Tomorrow we can talk about your work on our new carpet!"*

### Assessing Students' Learning

During this discussion, the teacher continues to assess students' work by looking and listening for:

- evidence of the problem-solving strategies that students are using;
- representations and justification of their solutions;
- understanding of concepts about area and perimeter;
- appropriate use of mathematical language.

The assessment information that the teacher gathers is used to plan next steps and future learning experiences.

### Home Connection

Sending the students home with a "job" serves several purposes and offers various benefits:

- Parents/caregivers are informed about what the students are learning.
- Parents/caregivers are involved in their child's math learning.
- Students have the experience of explaining the processes and strategies they used to solve the problem.
- Students feel as though they are doing math for a purpose.



## THE TEACHER'S ROLE IN TEACHING THROUGH PROBLEM SOLVING

As demonstrated in the problem-solving vignettes, the teacher's role is crucial in providing for students an effective problem-solving experience. Functioning as a problem-solving facilitator, the teacher helps students by:

- providing appropriate and challenging problems;
- supporting and extending student learning;
- encouraging and accepting students' own problem-solving strategies;
- questioning and prompting students;
- using think-alouds to model how a problem is tackled;
- observing students and assessing their work as they solve problems.
- anticipating conceptual stumbling blocks, noticing students who encounter these blocks, and helping them recognize and address their misconceptions.

Each of the first five of these aspects of the teacher's role in teaching through problem solving is developed in the subsections that follow. Observing students and assessing their work as they solve problems is treated in a separate section (see pages 48–50).

### Providing Appropriate and Challenging Problems

A good instructional problem captures students' interest and imagination and satisfies the following criteria:

- The solution is not immediately obvious.
- The problem provides a learning situation related to a key concept or big idea.
- The context of the problem is meaningful to students.
- There may be more than one solution.
- The problem promotes the use of one or more strategies.
- The situation requires decision making above and beyond the choosing of a mathematical operation.
- The solution time is reasonable.
- The situation may encourage collaboration in seeking solutions.

Good problems encourage students to reason their way to a solution and can result in one answer or in multiple answers, depending on the purpose and context. Standard textbook problems sometimes focus merely on translating a written word problem into a number sentence and using one specific method to find one specific answer. It is difficult to gain access to or improve higher-level thinking skills in students if only standard textbook problems are used in the classroom. Good problems focus students' attention on the underlying concepts of the big ideas – for example, as described in the companion documents to this guide that focus on the individual strands.



For instance, a student may be able to recall the answer to a question such as  $4 + 4$  but be unable to respond to a problem (e.g., "How many ways can we find to arrange these 8 gerbils in these two cages?") that entails recognizing the various addition combinations that make 8. In fact, students in the early stages of learning addition (without concrete experiences) might argue that  $3 + 5$  could not have the same answer as  $4 + 4$ . Standard textbook problems often focus solely on procedural knowledge, whereas good problems focus more on conceptual understanding. Good problems value the processes rather than just the end result.

Standard textbook problems are sometimes appropriate, and by opening them up so that a variety of solution strategies are possible, the teacher can elicit information about:

- how students reason;
- what strategies they have developed about mathematical reasoning;
- their use of a variety of representations;
- the efficiency and accuracy of their strategies.

As well, a good problem engages students' interest and allows students to use strategies that they have found personally useful rather than procedurally dictated. A good problem would allow students to make responses that reflect their level of understanding and reasoning and provide teachers with insight into their thinking.

The following four-step process can be helpful in generating good open-ended problems:

- Step 1: Identify the concept (rather than a skill or a procedure) that you want students to develop.
- Step 2: Locate the lesson or task in textbooks or professional resources that address the concept.
- Step 3: Ask the following questions about the lessons or task:
- Does the problem focus on conceptual understanding rather than just on the procedures?
  - Is the mathematics the central focus of the lesson or task?
  - Does the lesson or task require justifications and explanations for answers or methods?
  - Is there more than one method for arriving at a solution?
- Step 4: If the problem does not meet the criteria in step 3, revise the problem.

There are ways in which the teacher can revise a problem to make it a good problem and encourage students to become sense makers of mathematics. The table on the next page provides some examples of how problems can be opened up.

*"Developing mathematical power involves more than simply giving students harder problems. It means asking them to focus on understanding and explaining what they are doing, digging deeper for reasons, and developing the ability to know whether they can do a better job on working on a task."*

(Stenmark & Bush, 2001, p. 4)

## Revising Problems

Strand from the Ontario Curriculum	Initial Problems	Revised Problems
Number Sense and Numeration	$10 - 8 =$	The difference between two numbers is 2. What might the two numbers be?
	$10 \times 8 =$	The answer is 80. What might the factors be?
Geometry and Spatial Sense	How many sides does a triangle have?	Sort these two-dimensional shapes. Explain your sorting rule.
	What is a trapezoid?	Construct different quadrilaterals on a geoboard or on dot paper. Identify and describe each shape.
Measurement	What is the area of your desktop?	What different things can you find with an area of approximately two square metres?
	Calculate the perimeter of this rectangle.	Use a pencil and a ruler to draw different rectangles that have a perimeter of 24 cm. Do all the rectangles have the same area?
Patterning and Algebra	Make an AB AB AB pattern with the blocks.	Make a pattern, describe it, and tell me why you think it is a pattern.
	Show the multiples of 3 on the hundreds chart.	Find the patterns on the hundreds chart for multiples of 3 and of 6. Describe the similarities in the two patterns. Why do you think that this is so? Can you think of other multiples that will have intersecting patterns? (Possible answers: 4 and 8, 6 and 12.)
Data Management and Probability	Look at our bar graph about pets. How many students have dogs at home?	Remember our bar graph about pets and our other bar graph about how many brothers and sisters we have. I have taken off the titles of the graphs. How can you determine which graph would be the "pets" graph and which one would be the "brothers and sisters" graph? Why do you think this? (Possible answers: there are more categories for the "pets" graph; there are bigger numbers in the "brothers and sisters" graph.)
	Look at these graphs. Do primary students have more pets than junior students?	How could this graph help the owner of a pet shop sell more pets? (Possible answers: Children like dogs better than cats; boys are likelier than girls to own a pet.)

The parameters of the problems for Number Sense and Numeration were expanded by removing the specific numbers. For example, the revised subtraction question allows students to explore and communicate their understanding of the concept of the difference of 2, which is applicable to many number sentences, rather than to focus solely on answering the subtraction question  $10 - 8$ , which may only be an indication that students have memorized the computation without understanding. The revised multiplication question provides an open-ended challenge. Rather than simply giving an answer for  $10 \times 8$ , students investigate combinations of factors that produce 80.

The revised Measurement problems provide teachers with an opportunity to observe the depth of understanding that students possess in relation to the concepts of area and perimeter. Teachers will also discover whether students have merely memorized formulas but do not know how to apply them in novel situations. The second task also allows students to investigate the relationships between area and perimeter.

The revised problems for Geometry and Spatial Sense ask students to analyse and describe two-dimensional shapes, rather than simply to define shapes.

The revised examples for Patterning and Algebra encourage students to analyse and describe patterns. In solving these problems, students reveal their understanding of the meaning of the concept of pattern.

The final examples of revisions (Data Management and Probability) ask students to interpret graphs and extrapolate information rather than simply to find data on a graph. The revised problems also require students to justify their solution and explain their reasoning.

By consistently applying the criteria set out in step 3 on page 27, teachers can learn to be discriminating in the problems they use with students. They need not invent a new problem for every math lesson. By making small revisions to traditional problems found in older resources, teachers can make good problems that focus on the big ideas and key concepts. Such problems will ensure that students develop a deeper understanding of mathematics.

### **Supporting and Extending Learning**

The amount of support that the teacher provides throughout the problem-solving process must be finely tempered in response to the needs of students. A critical skill for teachers is recognizing how and when to prompt students to new levels of learning without dominating the learning experience. Students who arrive at solutions themselves learn valuable reasoning skills that can be applied in new situations. They make connections with past learning, and they develop a conceptual understanding

## Teacher Support During Problem Solving

### Getting Started

During this phase the teacher may:

- engage students in the problem-solving situation (e.g., personalize the problem by using students' names, or by putting himself or herself into the situation);
- discuss the situation;
- ensure that students understand the problem;
- ask students to restate the problem in their own words;
- ask students what it is they need to find out;
- allow students to ask questions (e.g., ask, "Do you need any other information?");
- encourage students to make connections with their prior knowledge (e.g., with similar problems or a connected concept);
- model think-alouds (this can also be done during and after the problem-solving activity);
- provide any requested materials and have manipulatives available.

### Working on It

During this phase, the teacher may:

- encourage brainstorming;
- use probing questions (see "Questioning and Prompting Students", on pp. 32–33);
- guide the experience (give hints, not solutions);
- clarify mathematical misconceptions;
- redirect the group through questioning, when necessary (students can usually recognize their own error if the teacher uses strategic questioning);
- answer student questions but avoid providing a solution to the problem;
- observe and assess (to determine next instructional steps);
- reconvene the whole group if significant questions arise;
- if a collaborating group experiences difficulty, join the group as a participating member;
- encourage students to clarify ideas and to pose questions to other students;
- give groups a five- or ten-minute warning before bringing them back to the whole-group discussion that takes place in the "Reflecting and Connecting" phase.

### Reflecting and Connecting

During this phase, the teacher may:

- bring students back together to share and analyse solutions;
- be open to a variety of solution strategies;
- ensure that the actual mathematical concepts are drawn out of the problem;
- highlight the big ideas and key concepts;
- expect students to defend their procedures and justify their answers;
- share only strategies that students can explain;
- foster autonomy by allowing students to evaluate the solutions and strategies;
- use a variety of concrete, pictorial, and numerical representations to demonstrate a problem solution;
- clarify misunderstandings;
- relate the strategies and solutions to similar types of problems to help students generalize the concepts;
- encourage students to consider what made a problem hard or easy (e.g., too many details, math vocabulary), and think about ways to achieve the appropriate degree of difficulty the next time;
- always summarize the discussion for everyone, and emphasize the key points or concepts.

that far exceeds the limits of strictly procedural knowledge. Teachers cannot give students that understanding; students need to develop it themselves. They do this with the support of the teacher and the teaching environment. The urge to provide the solution or reduce the process to rote procedures can be very strong. Students often pressure their teacher to give them the answer and to tell them how to solve the problem. Thus begins a cycle that continues from grade to grade and prevents students from developing the skills in perseverance that they need to be autonomous problem solvers. Teachers and students who have experienced the satisfaction that comes from persevering in the problem-solving process become better problem solvers. The teacher provides skilful questioning, thoughtful prodding, and appropriate modelling while students work through the problem-solving process.

The teacher's activities will vary throughout the problem-based lesson, depending on the phase of the lesson. The chart on the opposite page indicates the different ways in which the teacher supports students' learning during the main phases of the three-part lesson plan.

### **Encouraging and Accepting Students' Strategies**

Important to students' success in mastering problem-solving processes is the teacher's noncommittal acceptance of students' strategies. By providing questions that allow for different entry points into the problem-solving process, allowing a multitude of solutions, promoting student talk about the strategies, and engaging in thoughtful questioning, the teacher ensures that all students are valued as learners and that students gain an understanding of their own thinking and that of others. Students hone their reasoning skills as they listen to the reasoning of others and respond to probing by the teacher. Such listening helps students to recognize their misconceptions and adapt their strategies.

*"Discussions provide students with the opportunity to ask questions, make conjectures, share and clarify ideas, suggest and compare strategies, and explain their reasoning. As they discuss ideas with their peers, students learn to discriminate between effective and ineffective strategies for problem solving."*

(Ontario Ministry of Education ,  
2005, p. 25)

All problem-solving experiences should involve a "getting back together" stage, in which students gather, either in small groups or as a whole class, and the teacher ensures that everyone has developed a strategy for getting at the solution and has understood the math embedded in the problem. The "getting back together" stage can occur while students are solving the problem as well as at the end of the process.

One of the greatest contributions a teacher can make during the problem-solving activity is to step back and resist the impulse to give students the answer or to be overly prescriptive in how the problem should be solved. However, the teacher also needs to be sensitive to a student's level of frustration and to provide such support as is needed to ensure that the task does not become overwhelming. This can often be done through several well-chosen questions or by subtle redirection.

## Questioning and Prompting Students

Prompts, including the questions posed by the teacher as students interact, play a critical role in providing guidance and eliciting deeper thinking and reasoning. Questions and prompts can help the teacher to scaffold or provide support for students' learning. Through skilful questioning the teacher can subtly redirect a student's thinking so that he or she is successful in finding his or her own solutions and making sense of the mathematics.

There is a fine line between a question that encourages the student to think and one that provides the student with too much information or inadvertently solves the problem for the student. Being able to straddle this fine line comes with reflective practice.

It is always important for teachers to take note of the impact of their questioning on students' thinking. Teachers need to be careful not to halt students' thinking. Wait time is important in this regard. Allowing students time to struggle with a problem provides them with a valuable learning experience, but the teacher must be able to recognize a student's frustration and to address it by providing the question or prompt that will help unlock the student's thinking.

At various times throughout the math program, questions and prompts are used with different purposes in mind, for example:

- to help students retell
- to help students predict, invent, and problem solve
- to help students make connections
- to help students share their representations of mathematical situations
- to help students reflect on their work
- to help students share their feelings, attitudes, or beliefs about mathematics

Examples of the questions and prompts that facilitate these purposes are provided in the section "Questions and Prompts for Promoting Communication" in Chapter 6: Communication.

Although the teacher generates many questions and prompts while "in the moment" of working with students, he or she is wise to have planned a selection beforehand. The teacher can choose a few of the questions and prompts that relate to the purpose of the lesson, record them on a piece of paper, and attach the paper to a clipboard for easy reference.

Teachers develop effective questioning strategies over time with a great deal of practice. In developing their techniques, teachers will:

- *"rely on questions that require understanding and prompt thought rather than factual recall"* (Baroody, 1998, p. 17-8);  
By using verbs in prompts and questions (e.g., *explain, describe, apply, connect, elaborate, justify*), teachers ensure that students are more likely to communicate their reasoning and understanding.
- *"avoid yes-no questions"* (Baroody, 1998, p. 17-8);  
Yes-no questions (e.g., "Does a square have four sides?") encourage students to guess rather than to reason their way to a solution. A more open-ended question (e.g., "What do you notice about this square?"), on the other hand, encourages students to apply their understanding of the geometric properties of the shape.
- *"avoid leading questions"* (Baroody, 1998, p. 17-8);  
Rhetorical questions (e.g., "Doesn't a square have four sides?") provide students with an answer without allowing them to engage their own reasoning.
- *"avoid teacher-centered questions"* (Baroody, 1998, p. 17-8);  
For students, the purpose of questions is not that the teacher be given an answer but that a mathematical dialogue be opened up within the class. How questions are phrased will help students realize that their mathematical ideas are meant to be discussed and evaluated by all members of the class and not just by the teacher.
- *"avoid labeling questions as easy or hard"* (Baroody, 1998, p. 17-8);  
Qualifiers such as *easy* or *hard* can shut down learning in students. Some students are fearful of hard questions; others are disdainful of easy questions.
- *"avoid verbal and non-verbal clues"* (Baroody, 1998, p. 17-8);  
Verbal and non-verbal clues, such as facial expressions, gestures, and tones of voice, can limit learning. Students notice their teacher's emphasis on certain words, movement of the eyes, stiffening of the body – clues that signal the correct answer and make it superfluous for students to think things through for themselves. Rather than think about the question, students spend their time looking for clues that will give them the answer.
- *"regularly use wait time"* (Baroody, 1998, p. 17-8);  
Teachers who provide a wait time of three seconds or more after asking a question tend to be rewarded with a greater quantity and quality of student responses.

A good starting point for teachers who want to enhance their questioning skills is to begin to ask students, on a regular basis, the simple question "How do you know?". This question encourages students to think about their answers and the processes they used to arrive at a solution.



## Using Think-Alouds

Teachers can help students to expand their repertoire of problem-solving strategies by modelling the problem-solving process for them through think-alouds.

In think-alouds, teachers verbalize the thinking and decision making that takes place in their own mind as they work through a problem. For instance, they may demonstrate how to restate the problem in their own words and how to begin the problem by choosing a manipulative or picture to represent it in some visual or concrete manner. Then, they demonstrate the questions they would ask themselves as they try to determine a solution – for example: “What should I do first?” “How can I organize the information?” “What does this remind me of?” “If I use smaller numbers, will the problem be easier to manage?” Since the thinking processes involved in solving problems cannot be anticipated and planned for, teachers may find themselves retracting a previous statement or changing their approach as they think aloud. Such redirections and changes made by the teacher should be considered part of the learning process for students. Seeing teachers struggling at times helps students recognize that perseverance and persistence are important qualities in problem solving.

Critical to the teacher’s demonstration is an emphasis on the importance of reflecting at each stage of the process. Students often impulsively grasp the first strategy that comes to mind. By listening to teachers act out a scenario in which they think about their initial strategy and use some self-questioning, students begin to see the benefit of reflection. In developing their ability to reflect, students develop metacognition – that is, the ability to think about their own thinking. Eventually, they also develop the ability to articulate this self-reflection and can themselves engage in think-alouds as they work through problems.

On occasion, teachers may think aloud about the different strategies that were used and why some did not work as well as others. This context provides an opportunity for teachers to use the appropriate mathematical language for the task, modelling the language so that students can adopt it as their own.

## Teaching About Problem Solving

Although students are natural problem solvers, they benefit from guidance in organizing their thinking and approaching new problem-solving situations. Teaching about problem solving focuses on having students explore and develop problem-solving strategies and processes. Teaching about problem solving allows students and teachers to create strategies collaboratively and, at all stages of the problem-solving process, to discuss, informally and formally, the thinking and

*“Becoming a better problem solver is a gradual building process that requires taking on challenging and sometimes frustrating problems.”*

(Baroody, 1998, p. 2-11)



reasoning that they use in determining a solution. When teaching about problem solving, teachers provide students with opportunities to solve interesting and challenging problems.

In many cases, teaching about problem solving occurs simultaneously with teaching through problem solving. As students are engaged in a problem that focuses on a mathematical concept (as described earlier in the chapter), the skilled teacher integrates the discussion of problem-solving strategies and processes into the discussion of the mathematical concept. For some students, this approach may not be sufficient; such students may require additional and more focused opportunities to learn about problem solving. Whether the approach to learning about problem solving is an integrated or a more isolated one, teachers should ensure that, in their instruction about problem solving, they encourage students to develop their own ways of solving problems. To find their own ways, students must be aware of the variety of possible ways. They learn about new strategies by hearing and seeing the strategies developed by their peers, and by discussing the merits of those strategies.

In the pages that follow, a number of topics are explored in connection with teaching about problem solving: identifying classroom structures that provide opportunities for students to work on problem solving; recognizing the role and limitations of a specific problem-solving model; facilitating the exploration and sharing of student-generated problem-solving strategies; examining the teacher's role in teaching about problem solving; and considering ways to have students become problem posers.

## CLASSROOM STRUCTURES THAT SUPPORT PROBLEM SOLVING

When students are taught mathematical concepts through problem solving, they are exposed to problems and problem solving on a daily basis, even though the problem solving may not be formally identified as such. When students are taught about problem solving, they see problems and processes identified and named as they experience situations whose forms move beyond the "problem of the week".

*"A positive classroom climate is essential to build children's confidence in their ability to solve problems."*

(Payne, 1990, p. 41)

Teachers can provide a variety of classroom opportunities when teaching about problem solving. Such opportunities include:

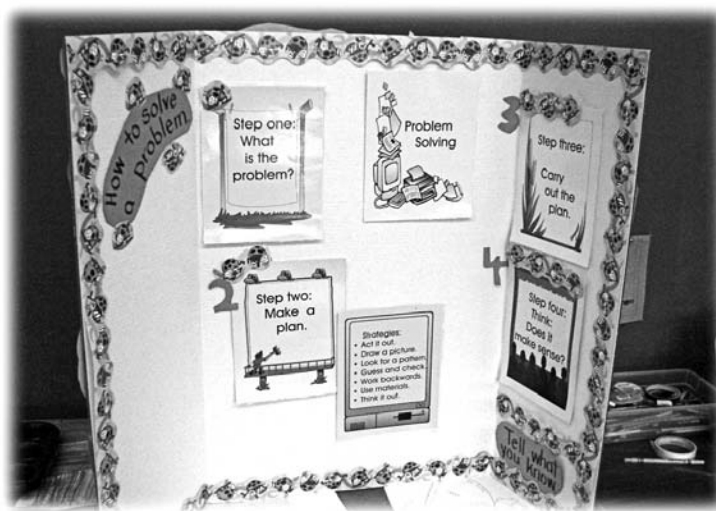
- **daily challenges** that can be utilized to engage students in regular problem solving. Teachers provide students with a meaningful problem to solve, either during a specific time of the day or whenever they have an opportunity throughout the day. Some teachers provide a challenge every day, or every other day, to engage students when they arrive in class first thing in the morning;

- **a problem-solving corner or bulletin board** that provides a place in the classroom where interesting problems can be posted. Students are given time throughout the day or the week to visit the corner and solve the problem. At some point, the whole class is brought together to discuss the problem, share their strategies, hear and see strategies used by other students, evaluate solutions, and at times cooperatively solve the problem;
- **an activity centre** that can be included as part of a rotation of centres in which students participate during the consolidation phase of a unit. The teacher provides a problem for students to solve collaboratively (e.g., with a partner or group). During class sharing sessions, the group from the problem-solving centre is asked to share its strategies. Students who have previously worked at the centre are asked to make comparisons with the strategies they used. Students may consider whether one strategy works better than another. (See also the subsection "Mathematics Learning Centres, Grades 1–6" in Chapter 7: Classroom Resources and Management, in Volume Three.)

## THE FOUR-STEP PROBLEM-SOLVING MODEL

One of the reasons to teach about problem solving is to help students develop a mental model of how to approach and persist with a problem-solving task. Students who may have good conceptual knowledge of mathematics may still have difficulty applying such knowledge in problem-solving activities, because they have not yet internalized a model that will guide them through the process. The most commonly used problem-solving model is Polya's four-step model – understand the problem, make a plan, carry out the plan, and look back to check the results (Polya, 1945).

Generally this model is not taught directly before Grade 3. Specific, directed lessons on the four-step model are not appropriate for young students, mainly because young students become too focused on the model and less concerned or connected with the mathematical concepts and sense making of the problem. But a teacher who is aware of the model and uses it to guide his or her questioning and prompting during the problem-solving process will provide students with a valuable skill that is generalizable to other problem-solving situations, not only in mathematics but in other subjects as well. The four-step model provides a framework for helping students to think about the question before, during, and after the problem-solving experience.



**The four-step problem-solving model is not generally taught before Grade 3.**

By Grade 3, the teacher can present the problem-solving model more explicitly, building on students' prior experiences in the previous grades. At this point and throughout the junior years, teachers can display the four-step model in the classroom and encourage students to refer to it for help in reflecting on the process used to find solutions. Teachers should communicate the idea that problem solvers do not always follow the stages of the model in a lockstep fashion; they often need to go back and forth between stages in order to understand the problem, try out strategies, and find appropriate solutions.

The phases of the four-step model are described below, together with some of the implications for teaching each phase.

*"Polya's model can also be misleading if taken at face value. Except for simple problems, it is rarely possible to take the steps in sequence. Students who believe they can proceed one step at a time may find themselves as confused as if they had no model."*

(Reys, Lindquist, Lambdin, Suydam, & Smith, 2001, p. 95)

### Four-Step Problem-Solving Model

#### Phases of the Four-Step Model

- Understanding the problem
- Making a plan
- Carrying out the plan
- Looking back and reflecting on the solution

#### Implications for Teaching

Students should be encouraged to think and talk about the problem and to restate it in their own words before they go to manipulatives or to paper and pencil.

Students should be guided to develop a plan. They should realize that all plans are tentative and may be changed throughout the process. They can consider strategies they might use. Suggestions such as looking at the classroom strategy wall might be helpful. It is not necessary for students to record the plan in writing.

At this phase, students are carrying out their plan and using strategies such as drawing a picture or working with manipulatives. Teacher prompting at this time should focus on questions that elicit greater understanding but should avoid inadvertently solving the problem for the student.

Perseverance at this stage should be encouraged. Suggestions to help the student become unstuck can be provided – for example: "Ask Natalie for an idea." "Refer to the strategy wall for another approach." "Can you think of a problem that is similar to this?"

During this "getting back together" phase, it is crucial that students share their ideas in the large group. As a result of the sharing, they can begin to discern that a variety of strategies can be used. They also begin to evaluate critically which strategy works best for them (e.g., is most efficient, is easiest to understand). Teachers should encourage students to discuss what they have learned through the problem-solving experience and to pose new problems that are related to the one just solved.

These phases should be used flexibly. For example, a student who has devised a plan to solve a problem may realize while carrying out the plan that it needs revision and will try something different. Or again, a student may realize at the end of the process that he or she needs to go back and try a different strategy in order to solve the problem. It is important for teachers to remember that the primary goal of problem solving is making sense of the mathematics rather than mastering the steps of a problem-solving model or a set of problem-solving strategies. To promote this goal, teachers will ensure that students have the blocks of time they need to see a problem through to a solution and to build perseverance and persistence.

*"Moreover, the steps are not discrete; nor is it always necessary to take each step."*

(Reys et al., 2001, p. 95)

*"Students must become aware of these strategies as the need for them arises, and as they are modeled during classroom activities, the teacher should encourage students to take note of them. For example, after a student has shared a solution and how it was obtained, the teacher may identify the strategy by saying, 'It sounds like you made an organized list to find the solution. Did anyone solve the problem in a different way?'"*

(NCTM, 2000, p. 54)

## PROBLEM-SOLVING STRATEGIES

Problem-solving strategies are specific methods used for solving problems. These strategies are best explored by students incidentally, within the context of solving daily problems, rather than through direct instruction about the strategies themselves. Direct instruction in the strategies is less likely to produce good problem solvers who are able to use problem-solving strategies flexibly and who achieve the "ultimate aim of independent or autonomous problem solving" (Baroody, 1998, p. 2-15). Good problem solvers emerge as they work with their peers on specific problems and share and reflect on the variety of strategies they generated or used to solve problems. As students learn how different strategies work, they become adept at choosing the strategies that are most effective and efficient in solving problems in new situations.

*"Strategies are not learned at a specific time or in a single lesson. Children will use them when they are ready. We structure situations that promote their use, but realize that the child has to decide to use them."*

(Trafton & Thiessen, 1999, p. 44)

Teachers can assist students in a number of ways as they explore problem-solving strategies within the context of solving daily problems. For example, if students in their sharing have not mentioned a certain appropriate strategy (e.g., make a model) as one that they have used, the teacher can guide discussion by asking questions such as the following:

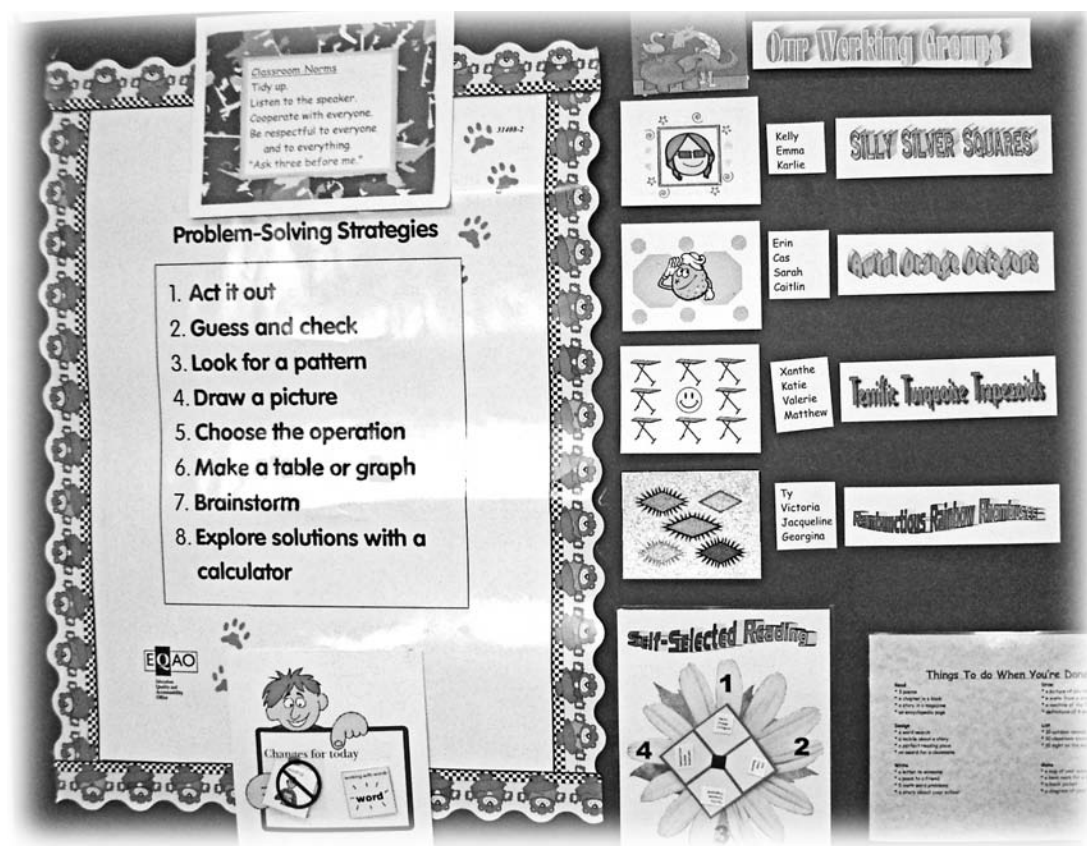
- "Did anyone use a manipulative to help them solve the problem?"



- "How could a manipulative help us?"
- "Which manipulative would help you to solve the problem?"

Teachers can also model strategies during shared problem-solving situations in the classroom, either while students are actually in the process of solving the problem or later, during the discussion that takes place after the problem-solving activity has been completed. For example, the teacher could draw a picture to represent a particular problem if he or she had observed that students were not using this strategy.

Some of the strategies that students may employ while involved in various problem-solving tasks are listed in the pages that follow. These tend to be the most frequent strategies that students develop or can be guided to develop, but the list is not exhaustive and not all strategies will be developed or utilized every year, in every grade. It takes many opportunities of guided, shared, and independent problem solving for students to develop a repertoire of problem-solving strategies. Throughout the primary and junior years (Junior Kindergarten to Grade 6) many of the strategies in the table will be generated by students naturally when solving problems. The value of student-developed strategies is that they use the student's own words and they are internally connected with the student's understanding.



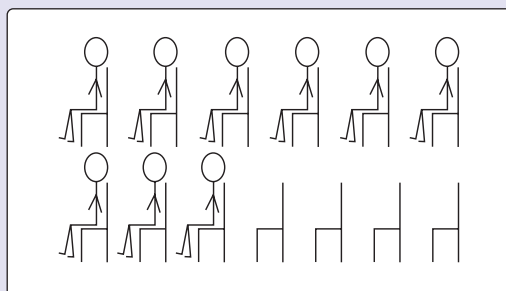
**Act It Out****1-3****4-6**

With this strategy, students act out a problem situation in order to find a solution. For most students, acting out the problem extends their understanding of it into a real situation.

For example, students may be asked to solve the following problem:

The school bus picks up many children on the way to school. The bus makes three stops altogether. At the first stop, 4 children get on the bus. At the next stop, 5 more children get on the bus. By the time the bus arrives at school, there are 13 children on the bus. How many more children got on the bus at the third stop?

They may “act it out” by lining up 13 chairs in the classroom and modelling each addition of the children onto the bus. Once all the children who boarded the bus at the first two stops have taken their seats, the students can solve the problem by counting the empty seats.

**Make a Model With Concrete Materials****1-3****4-6**

This strategy is similar to the strategy of acting it out. However, manipulatives or materials are used to represent aspects of the problem, instead of physical actions.

For example, students may be asked to solve the following problem:

The teacher has 16 stickers to give to 4 students. How many stickers will each student get?

They may use manipulatives or materials (e.g., beans, bingo chips, bottle caps, colour tiles) to model their understanding of the problem by creating 4 piles, with 4 items in each pile.

Junior students continue to benefit from the use of concrete materials to represent and visualize new or developing concepts. For this reason, teachers should ensure that manipulative materials are readily available, and should encourage students to use them to represent mathematical ideas.

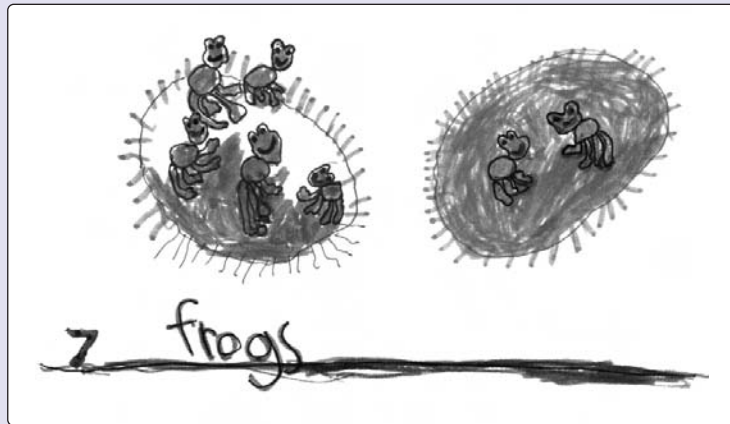
**Draw a Diagram****1-3****4-6**

This strategy helps students understand the problem as well as solve it. Drawing a picture to illustrate the action in a problem assists students in making the problem real for them. Often if students use concrete materials to solve a problem, drawing a diagram to represent the use of manipulatives usually follows. Eventually, students rely less on the use of concrete materials and move more towards using pictorial representations.

For example, students may be asked to solve the following problem:

Najeeb and his family went to the pond. They watched the frogs for a long time. Najeeb saw 3 frogs sitting on one lily pad and 2 frogs on another lily pad. Two more frogs jumped onto the first lily pad. How many frogs are there altogether?

Then 4 frogs swam away. How many frogs are there altogether now?



They may draw a picture of the two lily pads. Students will begin by drawing 3 frogs on one lily pad and 2 frogs on the other lily pad. Students will then draw an additional 2 frogs on the first lily pad. To solve the second problem, they will cross out 4 frogs.

Junior students continue to benefit from using visual representations. For example, drawing diagrams is an effective strategy to use in solving problems in which objects are divided into fractional parts.

### Use the Guess-and-Check Method

**1-3** **4-6**

This strategy allows students to experiment with their understanding of problems and with making “good” guesses. Students need to be encouraged to try out different guesses, and then make better guesses based on prior knowledge and experience. Students need to appreciate that guessing is a valid and useful strategy when tackling a problem. As students develop this strategy, they realize that considering the information provided in a problem and estimating what might be a reasonable solution are steps that assist them in making educated or “good” guesses. Students should be encouraged to record their guesses as they solve the problem, to help them refine their guessing skills.

For example, students may be asked to solve the following problem:

I have a total of 48 cents. I have twice as many pennies as nickels and twice as many nickels as dimes. What coins do I have?

They may use diagrams or concrete materials to help them while using a guess-and-check method. Students should be encouraged to record their guesses as they solve this type of problem.

It is also extremely important for students to see that “wrong” answers or “incorrect” guesses are an integral part of the problem-solving process. Inaccuracies are often the stimulus for mathematical discussions and discoveries.



## Work Backwards

**1-3****4-6**

With this strategy, the problem solver needs to consider the final condition of an action and to determine what occurred earlier.

For example, students may be asked to solve the following problem:

Mary, Lucas, and Nabil collect hockey cards. Mary has the most cards. She has 12 more cards than Lucas does. Lucas has 7 more cards than Nabil has. Nabil has only 10 cards. How many cards does Mary have?

They may begin with 10, the number of cards that Nabil has, and add 7 in order to calculate the number of cards that Lucas has. To this sum (i.e., 17), students would add 12, to find that Mary has 29 cards.

## Use Logical Thinking

**1-3****4-6**

This strategy provides students with opportunities to use logical thinking as they explore several types of problems. One logic-type problem encourages students to analyse information, use clues, and use deductive reasoning to arrive at a solution. Another logic-type problem encourages students to discriminate among attribute blocks by observing the various attributes (e.g., shape, size, colour) of the blocks.

For example, students may be asked to solve the following problem:

Four children made posters and displayed them. Kareem's poster is next to Sarah's. Brent's poster is on the far left. Elan's is on the far right and is next to Kareem's. How are the posters arranged? Give the order from left to right.

They may use a diagram or concrete materials, as well as the guess-and-check method, to explore this problem. Students will list the posters from left to right, following the clues provided in the problem.

## Make a Table

**1-3****4-6**

This strategy assists students when they are examining number relationships, patterns, or data. The strategy is frequently used in combination with using concrete materials, drawing a diagram, or acting out a problem.

For example, students may be asked to solve the following problem:

On the first day of school the teacher put 1 penny into a jar. On the second day of school he put 2 pennies in the jar. On the third day of school he put 4 pennies in the jar. Every day the teacher put twice as many pennies in the jar as the day before. How many pennies would be in the jar on the fifth day of school?

They may use a T-chart to help organize the information and then extend the pattern. This strategy allows students to see the relationship between the number of days and the number of pennies, and then to find the total number of pennies in the jar on the fifth day.

Days	Pennies
1	1
2	2
3	4
4	8
5	16



**1-3**      **4-6**

For example, students may be asked to solve the following problem:

They may create an organized list, as well as draw pictures or use manipulatives, to solve this problem. Students should find a way to organize their lists so that they will know when they have found all the possibilities.

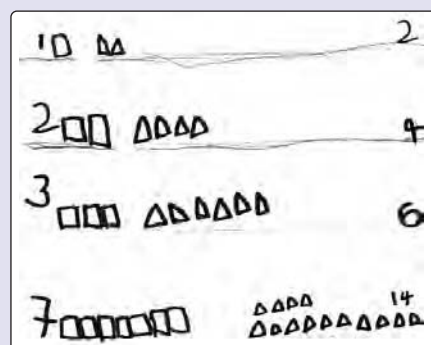
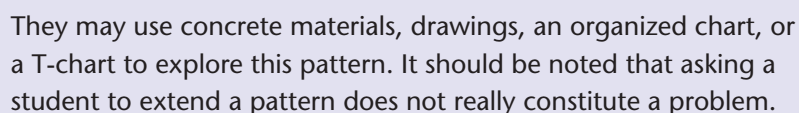
1-3 4-6

For example, students may be asked to solve the following problem:

Students can think of a simpler, but similar, problem. For example, they might recognize that 2 buses would transport approximately 70 students. Since there are 10 times as many students (700 rather than 70), they would multiply 2 by 10 to determine that approximately 20 busloads of students will travel to the fair.

1-3 4-6

For example, students may be asked to solve the following problem:



Teachers should avoid teaching students to use key words as a strategy for solving word problems. When using this strategy, students underline, circle, or highlight key words in a problem to help them know what operation to use. For example, students are taught to look for such words as *together*, *total*, and *difference* when solving problems. This particular strategy encourages students to suspend their own mathematical sense making and instead to see the problem as a simple arithmetic question to complete. Overreliance on this strategy prevents students from making sense of the context of the problem and keeps them from looking for underlying assumptions or missing information.

A better strategy would be to have students discuss the known information, the unknown information, and the asked-for information (Van de Walle, 2001). Students need to make sense of the entire problem in order to be successful in handling increasingly complex problems as they move through the grades.

## THE TEACHER'S ROLE IN TEACHING ABOUT PROBLEM SOLVING

### Helping to Develop Strategies

The best strategies and models for problem solving are those that students develop themselves or that they learn from other students. The teacher's role is to help students articulate the strategies and then to keep an ongoing record of the strategies through the use of a strategy wall (the strategy wall is discussed more fully in the subsection "Strategy Walls" in Chapter 7: Classroom Resources and Management, in Volume Three) or any similar type of display that students can readily gain access to and refer to (e.g., a chart or a bulletin board). As students develop their understanding of concepts through solving problems, teachers ensure that students have opportunities to share the various strategies they use.

*"Helping students become good problem solvers is like helping them learn how to ride a bicycle; tips can be helpful, but it's impossible to master the process without actually trying it."*

(Baroody, 1998, p. 2-11)

Teachers guide this process of developing strategies with students by:

- providing problem-solving situations that are engaging and that genuinely require the use of problem-solving strategies;
- facilitating for students opportunities to discuss "what is known, needed, and asked for" when faced with problem-solving situations (Van de Walle, 2001, p. 57);
- allowing students to determine a way to solve the problem that makes sense to them rather than forcing them to use a particular strategy;
- facilitating for students opportunities to share the processes and steps they use and the thinking they do when they are solving a problem;

- providing for students opportunities to share with the class the strategies that they have developed for solving a particular problem;
- offering suggestions of ways to solve a problem when and if students become stuck;
- encouraging risk taking and being sensitive to students who take risks when sharing their personal strategies with partners or in a small group, and at times with the whole class;
- encouraging students to persevere in problem solving by providing sufficient time and discussion to build confidence;
- using prompts and questions to help students become unstuck (e.g., see "Questioning and Prompting Students", on pp. 32–33);
- modelling ways to organize data (e.g., by using charts), represent data (e.g., by using manipulatives), and share data (e.g., by talking or writing about the data) in shared problem-solving experiences;
- collaboratively building a class strategy wall as a public display of the strategies that students have created and shared throughout the year;
- encouraging students to try out strategies that other students create, and asking them to discuss, critique, and evaluate the efficiency of various strategies.

*"We want children to develop confidence in their ability to solve problems because the belief that they can solve problems will influence their performance."*

(Payne, 1990, p. 41)

## Choosing Problems

Students need to tackle challenging problems on a regular basis. Good problems capture students' interest and imagination and satisfy the following criteria:

- The solution is not immediately obvious.
- There may be more than one solution.
- The problem promotes the use of one or more strategies.
- The situation requires decision making above and beyond the choosing of a mathematical operation.
- The solution time is reasonable.
- The situation may encourage collaboration in seeking solutions.

Good problems can be found in a variety of professional resources, the Internet, and/or textbooks.

*"Making traditional word problems the only or main emphasis of the problems children encounter is not sufficient for the elementary curriculum. Doing so gives an unrealistic message to children about the way mathematics will serve them as adults. Most daily problems adults face that require mathematics reasoning and skills are not solved by translating the available information into arithmetic sentences and then performing the needed calculations."*

(Burns, 2000, p. 15)

## PROBLEM POSING

One way of helping students develop their understanding of problem solving is to encourage them to pose their own problems, either orally or in writing. In order to be able to pose a problem, students will need to know the mathematical concepts in the problem and be able to apply them in a problem-solving situation. Often, students reveal more about their understanding of a concept in posing problems than they do in solving teacher-initiated problems. For instance, students in Grade 3 who may seem to have a good understanding of multiplication often have significant difficulty in articulating a problem that involves using multiplication for its solution. In fact, the problems that Grade 3 students put forward often relate to addition (and not repeated addition) rather than multiplication. The following is an example of a multiplication problem posed by a Grade 3 student:

**1-3**

There are 3 dogs in my family, 2 cats in Ramiro's family, and 1 bird in Tom's family. How many pets are there in the three families?

Students in the junior grades often have difficulty with posing a problem involving fractions. For example, a Grade 5 student poses the following problem, which he believes provides an answer of  $\frac{1}{2}$ :

**4-6**

I need  $\frac{1}{4}$  cup of flour and  $\frac{2}{8}$  tablespoons of sugar for a recipe. How much do I need altogether?

The student's question reveals a fundamental misunderstanding of fractional relationships. The student does not recognize that a fraction, such as  $\frac{1}{4}$ , can represent a small or a large amount, depending on the size of the whole.

To help students become more proficient in problem posing, teachers can use a variety of activities and approaches, including the following:

1. The teacher allows students to take turns posing problems for the class to solve. The teacher sets up a schedule so that each student gets a turn.
2. The students make up a problem-solving game. Each student poses a problem on one side of an index card and records a solution on the other side. The cards are then shuffled and placed in a pile at the problem-solving centre. When students visit the problem-solving centre, they work in pairs. They select a problem to solve, develop a solution, and compare their strategy with the one suggested on the back of the card.

3. The teacher provides younger students with chart-paper prompts or photocopied sheets that help guide their problem posing. Each student can fill in the details, such as names and numbers, and give the problem to another student to solve. For example:

\_\_\_\_\_ ate \_\_\_\_\_ berries and

\_\_\_\_\_ ate \_\_\_\_\_ berries.

How many berries were eaten?

4. The teacher provides each student with a page from a toy catalogue and encourages each student to design a problem. For example, students might create the following problems:

I spent more than \$100 and less than \$150. What could I have bought?

I bought five items but spent less than \$100. What might I have bought?

5. The teacher gives students an answer and tells them they need to write a problem to go with the answer. For example, if the teacher says "My answer is 10 balloons," a student might write this problem:

Jean had 3 red balloons and 7 green balloons. How many balloons did she have?

6. The teacher uses a picture book as a stimulus to help students write problems in story format. For example, the book *12 Ways to Get to 11* by Eve Merriam (Simon and Schuster, 1993) might inspire a student to write his or her own "book" titled "4 Ways to Get to 5".

7. Students use a number as the impetus for creating a problem. For example, the teacher asks students to write a problem that requires the answer of  $\frac{1}{2}$  or the use of the number  $\frac{1}{2}$  somewhere within the problem.

As teachers initiate problem posing in the classroom, they should take into account that young students need to have oral experiences in developing problems before they move to paper-and-pencil tasks.

## Observing Students and Assessing Their Work as They Solve Problems

According to Baroody (1998), there are numerous characteristics that have an impact on a student's ability to solve genuine problems. Divided into categories, they are characteristics that involve:

- cognition
- affect
- metacognition
- flexibility

Teachers may find it helpful to think about their assessment practices in problem solving in relation to these four categories, and to use the categories as a basis for making observations and collecting data that will provide insights into students' problem-solving abilities, attitudes, and beliefs.

Characteristics involving cognition that affect student problem solving include the following:

- the ability to take existing information (e.g., strategies or processes) into a new situation and know how to use it;
- the adaptive expertise to use sense making and reasoning to solve a problem in a way that does not rely solely on memory, procedures, and rules.

Characteristics involving affect (or feelings and beliefs) that influence student problem solving include the following:

- a positive emotional response towards mathematics and problem solving;
- self-confidence as a problem solver;
- a perception of mathematics as something that can be of interest and of help in learning about the world;
- the ability to persevere in learning skills for coping in the face of a challenging problem;
- the ability to take risks and know that the mathematics class is a safe environment in which students' ideas are valued and their mathematical thinking, ideas, and/or strategies are neither ridiculed nor criticized;
- a belief that mistakes are a way of learning more and an opportunity to deepen and enhance understanding.

Characteristics involving metacognition that affect student problem solving include the following:

- the ability to think about one's own thinking;
- the ability to recognize that a solution makes sense and is reasonable;
- the possession of strategies for knowing what to do when one does not know what to do;
- the ability to self-monitor throughout the problem-solving process.

*"Thus, it is essential that beliefs, attitudes, and thinking processes be evaluated, as well."*

(Payne, 1990, p. 59)

Characteristics involving flexibility that affect student problem solving include the following:

- an understanding that the first plan is not necessarily the only plan and that plans are often modified throughout the process;
- an understanding that there is usually more than one way to arrive at a solution;
- an openness to the ideas of others;
- a willingness to try new ways or strategies;
- an understanding that some problems can be interpreted in more than one way.

Using the characteristics listed above, teachers can make observations about students and instructional decisions based on those observations. These observations occur incidentally as well as explicitly through all stages of the problem-solving process. The information that teachers gain about their students as they are actively engaged is valuable not only as a stimulus for on-the-spot minilessons but also as a factor in making decisions about the course of future instruction. For example, if a student is not achieving, the teacher can review the characteristics to consider whether the issue involves cognition (e.g., the student lacks the ability to apply a specific strategy within the context of the problem), affect (e.g., the student lacks confidence), metacognition (e.g., the student lacks the ability to monitor his or her thinking throughout the process), or flexibility (e.g., the student is rigid in his or her thinking and has a view that there is only one way to solve the problem). On the basis of these observations, the teacher can plan and initiate next instructional steps to help the student. In the chart on the following page, some suggestions are provided for responses that the teacher can make after observing students while they are problem solving.

*"Evidence of children's self-confidence or belief in themselves as problem solvers can be seen in such behaviors as being willing, even eager to try; persisting if first efforts fail; representing and solving some problems in more than one way; and realizing that some problems may have more than one answer."*

(Payne, 1990, p. 59)



Observation	Next Steps
<p><i>The teacher observes whether students:</i></p> <ul style="list-style-type: none"> <li>• make conceptual connections between mathematical ideas</li> </ul>	<p><i>The teacher needs to:</i></p> <ul style="list-style-type: none"> <li>• provide encouragement to students as they begin to make conceptual connections</li> <li>• provide prompts that help to scaffold and support students' developing ideas</li> </ul>
<ul style="list-style-type: none"> <li>• approach problem-solving situations with confidence</li> </ul>	<ul style="list-style-type: none"> <li>• avoid dominating the problem-solving situation</li> <li>• allow students adequate "think time"</li> <li>• allow each student to work with a partner to build confidence</li> <li>• maintain a positive attitude towards problem solving</li> </ul>
<ul style="list-style-type: none"> <li>• use self-monitoring strategies</li> </ul>	<ul style="list-style-type: none"> <li>• help students develop strategies for reasoning their way towards an answer (e.g., encourage students to use what they already know, such as making tens to help with addition)</li> </ul>
<ul style="list-style-type: none"> <li>• are flexible in using strategies and processes</li> </ul>	<ul style="list-style-type: none"> <li>• provide various students with opportunities to share their strategies for solving a problem in different ways</li> <li>• build a strategy wall as students in the class generate new strategies</li> <li>• remind students that there are many ways to solve problems</li> <li>• celebrate diversity in thinking</li> </ul>

Appendix 5-1 in this section provides an anecdotal tracking form that can be used throughout a term to record observations of individual students and note next instructional steps to take.

To sum up, teaching mathematics through problem solving should begin in the early years and continue to be the main focus in all grades. As students are engaged in exploring and developing mathematical concepts, the teacher highlights the processes and strategies of problem solving. Teachers become role models for problem solving by being flexible, by modelling a variety of strategies, and by encouraging students to use strategies that make sense to them. Finally, because attitudes and beliefs about problem solving have a major impact on student learning, the most important influence that a teacher can have on students is to help them develop attitudes and beliefs that confirm their capability as good problem solvers.



## Appendix 5-I: Problem-Solving Tracker

Student: \_\_\_\_\_

Observations	Next Steps
Characteristics involving cognition	
Characteristics involving affect	
Characteristics involving metacognition	
Characteristics involving flexibility	



# 6. Communication

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# Communication

## Introduction

### THE IMPORTANCE OF COMMUNICATION

The secret to successful teaching is being able to determine what students are thinking and then using that information as the basis for instruction. Teachers learn what students are thinking through student communication. When students communicate mathematically, either orally or in writing, they make their thinking and understanding clear to others as well as to themselves. In the early grades, students' thinking about mathematics is often difficult to explore, primarily because students' skills in talking and writing are just beginning to develop and because their experience of communicating about mathematics is new. Throughout the primary grades and into the junior years, however, students gain more experience and are given many opportunities to acquire an increasing number of strategies for demonstrating what they understand mathematically and demonstrating the process they used to find a solution.

Having students communicate mathematically helps teachers to:

- gauge students' attitudes towards mathematics;
- understand student learning, including misconceptions that students have;
- help students make sense of what they are learning;
- recognize and appreciate another perspective.

When communication is emphasized in the mathematics program, students also have many opportunities to develop and reinforce their literacy skills. In order to investigate mathematical concepts and solve mathematical problems, students need to read and interpret information, express their thoughts orally and in writing, listen to others, and think critically about ideas. Many of the communication strategies described in this chapter are not unique to mathematics learning – they are instructional techniques that can be used across the curriculum.

*“At the heart of mathematics is the process of setting up relationships and trying to prove these relationships mathematically in order to communicate them to others. Creativity is at the core of what mathematicians do.”*

(Fosnot & Dolk, 2001, p. 4)

*“Literacy learning becomes more meaningful when reading, writing, talking, listening, and thinking are integrated into subjects across the curriculum, and when students are encouraged to connect what they are learning in school with a growing awareness of the world beyond school.”*

(Expert Panel on Literacy in Grades 4 to 6 in Ontario, p. 27)

## WAYS OF FOSTERING COMMUNICATION

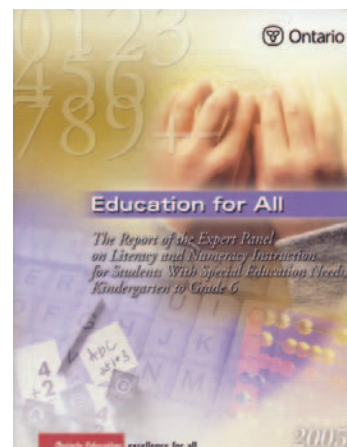
To understand what students are thinking about mathematics and to help them acquire skills in oral and written communication about mathematics, teachers will first and most importantly provide a sound basis of mathematical content – that is, a variety of rich problem-solving experiences – so that students will have something to talk or write about.

Teachers also engage students many times every day in classroom discussions. These discussions help the teacher to understand students, and they benefit students in many ways in learning mathematics. Through skilfully led discussion, students build understandings and consolidate their learning. Discussions provide students with the opportunity to ask questions of one another, to make conjectures, to share ideas, to test and clarify those ideas, to suggest strategies, and to explain their reasoning. In hearing the ideas of their peers, students learn, with the guidance of the teacher, to discriminate between effective and ineffective strategies for problem solving.

Students' writing will also reveal something of their understanding, but it is not necessary that all mathematics learning involve a written communication component. Young students need opportunities to focus on their oral communication without the additional responsibility of writing. Students in the junior grades are better able to handle the demands of explaining their reasoning in a written format, but even if these students possess skills in writing, their math learning need not always include a written task. Regardless of the age or grade, however, all students should be given the opportunity to verbalize their ideas before being asked to record them.

Whether students are talking or writing about their mathematical learning, the most valuable question the teacher can ask of them is "How do you know?" In answering this question, students must explain their thinking and the mathematical reasoning behind a solution or strategy.

Students with special education needs may require additional support in developing their mathematical communication skills. For information on the ways in which such support can be provided, refer to pages 85–90 in Chapter 7, "Effective Instructional Approaches and Teaching Strategies for Numeracy", in *Education for All: The Report of the Expert Panel on Literacy and Numeracy Instruction for Students With Special Education Needs, Kindergarten to Grade 6, 2005*.



## TWO APPROACHES TO PROMOTING COMMUNICATION

Because mathematical reasoning must be the major focus of students' communication, teachers should select an instructional strategy that elicits mathematical reasoning from their students. In particular, teachers can take two approaches: they can have students explain how they justify their own mathematical conclusions or solutions, or they can have students make and/or prove conjectures.

In order to understand the mathematics they are learning, all students go through a process in which they justify to themselves their conclusions about a concept or procedure. When they are asked to share that understanding, they must communicate in a way that convinces others that they are correct. This means that they must use valid arguments and precise mathematical language. For example, a student who is asked to explain how he or she knows that there are 100 squares on a hundreds chart without counting each square might say: "I know that each row has 10 squares because I counted one row, and all of the other rows are the same length. I know that each column has 10 squares because I counted one column, and all the other columns are the same height. I also know that  $10 \times 10$  is 100, so that is how I know there are 100 squares without counting them individually." Another student might have a different, yet equally valid, answer to the same question, and that student's understanding would be revealed in his or her explanation.

Having students engage in the process of making and proving a conjecture is another way of helping them understand a mathematical concept or procedure and communicating about it. For example, if the teacher makes the conjecture that multiplying two numbers always gives a larger number than adding those same two numbers and then asks students to agree or disagree, they will go through a process of proving or disproving the conjecture. In that process, they will come up with examples that make the statement true or make it false and will agree or disagree with the conjecture. A key question for engaging students in conjecture is this: "Is it true for all numbers?" A further step would be to have certain students come up with and share a conjecture of their own. Other students would be challenged to articulate arguments proving or disproving the conjecture. In the wording of both the original conjecture and the arguments for and against it, students would be required to use precise language. In this process of making and proving conjectures, students' understanding of the mathematical ideas would be deepened.

# Communication in the Classroom

## OPPORTUNITIES FOR COMMUNICATION

Fostering students' communication skills is an important part of the teacher's role in the mathematics classroom, and teachers need to be aware of the many kinds of opportunities that exist in the classroom for helping students to communicate. For example, teachers can:

- promote mathematical tasks that are worth talking about;
- model how to think aloud, and demonstrate how such thinking aloud is reflected in oral dialogue or in written, pictorial, or graphic representations;
- encourage students to think aloud. This process of talking should always precede a written strategy and should be an integral component of the conclusion of a lesson;
- model correct mathematics language forms (e.g., *line of symmetry*) and vocabulary;
- encourage talk at each stage of the problem-solving process. Students can talk with a partner, in a group, in the whole class, or with the teacher;
- ask good questions and encourage students to ask themselves those kinds of questions;
- ask students open-ended questions relating to specific topics or information;
- encourage students to ask questions and seek clarification when they are unsure or do not understand something;
- provide "wait time" after asking questions, to allow students time to formulate a response;
- pair an English language learner with a peer who speaks the same first language and also speaks English, and allow the students to converse about mathematical ideas in their first language;
- model the ways in which questions can be answered;
- make the language explicit by discussing and listing questions that help students think about and understand the mathematics they are using;
- give immediate feedback when students ask questions or provide explanations;
- encourage students to elaborate on their answer by saying, "Tell us more";
- ask if there is more than one solution, strategy, or explanation;
- ask the question "How do you know?"

*"Writing and talking are ways that learners can make their mathematical thinking visible."*

(Whitin & Whitin, 2000, p. 2)



In this chapter, a vignette similar to the one for problem solving in Chapter 5 provides one focus for looking at communication in the mathematics classroom. The vignette presents the first lesson in a series of lessons in which a Grade 4 class explores fractions in a fair-sharing context. In the vignette, communication is shown as a pervasive element in the mathematics classroom. Following the vignette, the chapter focuses on some specific strategies that teachers can use in promoting oral and written communication.

## WHAT COMMUNICATION IN THE CLASSROOM LOOKS LIKE

The following vignette helps to paint the picture of what communication looks like in a Grade 4 classroom. The lesson is structured in three phases – “Getting Started”, “Working on It”, and “Reflecting and Connecting”. The left-hand side of the page is used for telling the story of what the teacher and the students say and do during a problem-solving activity when the teacher is working to promote effective communication about mathematics among his or her students. The right-hand side of the page is used for describing the actions and decisions of the teacher throughout the process of solving the problem. The classroom description provides examples of how the teacher models good oral and written communication skills and gives students opportunities to use a full range of communication skills.

**Getting Started** (Understand the problem) (*Preparing for learning*)

The purpose of this vignette is to highlight how a teacher enriches the learning experiences of Grade 4 students by promoting effective communication.

In this investigation students explore fractions, using a fair-sharing situation. The hope is that this problem context will help students make the connection between fractions and division and will eventually lead them to explore the idea of equivalent fractions. This open type of investigation allows students with varying developmental levels to explore the problem successfully.

The teacher presents the following problem:

At Camp Kenabe, each cabin earned pizza on Pizza Day by completing a certain number of camp-ground chores. The Grizzly Bear cabin of 4 campers earned 3 pizzas. The Snowy Owl cabin of 5 campers earned 4 pizzas. The Caribou cabin of 10 campers earned 9 pizzas, and the Salmon cabin of 5 campers earned 3 pizzas. Some of the campers said that this was not fair – that some campers got more pizza than others. Were they right?

This is a quality investigation that will promote many opportunities for communication because it:

- involves the meaningful use of mathematics – students will identify with the problem context and will be interested in discussing a solution;
- allows learners of varying abilities to enter into solving the problem – if every student has an entry point into the investigation, more students will be likely to contribute to discussions;
- is rich enough to allow for adaptations, extensions, and connections – this richness allows for ongoing questioning, discussions, and discoveries;
- is collaborative – the student can interact with other students or the teacher in order to solve the problem;
- provides students with opportunities to explicitly demonstrate good communication skills.

The teacher asks the students to do a think-pair-share for five minutes to brainstorm and discuss some opening ideas about the problem, which may be shared later during the whole-class discussion.

After five minutes have passed, the teachers brings the students back to a whole-class discussion.

During the think-pair-share activity, the teacher circulates around the room, listening to the ideas of the students to:

- assess their prior knowledge;
- identify possible areas of difficulty.

The whole-class discussion is an opportunity for students to share their initial thoughts about the problem and to listen to the ideas of others.

The teacher asks, “What are you thinking? Is this fair?”

“I think it is fair,” offers Dahlia. “It is 1 less pizza than the number of kids in the cabin each time.”

“Let’s use our ‘thumbs up’ to see how many of you agree with Dahlia.” The teacher observes that there is a mixture of thumbs up and thumbs down in the class.

In the whole-class discussion, the teacher asks students to volunteer ideas that they were discussing during the think-pair-share. The students generate most of the discussion, with the teacher listening and adding prompts where necessary.

The thumbs-up/thumbs-down strategy gives students a quick and silent communication method and gives the teacher a quick view of students’ thinking.

*"Dylan, why do you disagree?"*

*"I don't think it is fair for the Salmon cabin. They had 5 kids and only 3 pizzas, so for them they have 2 less pizzas than the number of kids."*

*"Are the pizzas all the same size?" asks Jasmine.*

*"That's a good question. In this investigation, we can assume that the pizzas are all the same size."*

Shibhana asks, *"How will the pizzas be shared?"*

*"You could cut them into pieces like halves or fourths or fifths," suggests Alex.*

*"But then the pieces will be different sizes," Tana points out.*

*"How do you know that?" the teacher asks.*

Tana replies, *"I was at a birthday party once and there were two cakes that were the same size, but one was cut into more pieces than the other. I got a piece from one cake and my brother got a piece from the other cake. He was teasing me because his piece was bigger than mine."*

*"You've brought up some important ideas to remember when you are working on this problem. Can someone remind us about what we do know about this problem?"*

Natasha says, *"We know we have four different cabins that are getting pizza, but each cabin has different numbers of campers and of pizzas. We have to find out if all the campers get the same amount of pizza."*

*"This sounds like an interesting problem, Natasha. Do we have any information that will help us solve it?"*

Other students respond:

- *"We know the number of campers in each cabin and how many pizzas each cabin got."*
- *"We know that all the pizzas are the same size."*
- *"We know that if the pizzas are cut into a different number of pieces, the pieces might not be the same size."*

As the students share the known information, the teacher records it on a chart.

The discussion does not focus on who is right or wrong but on the problem. This creates a comfortable atmosphere for student sharing, even when there is disagreement.

Discussion helps students activate prior knowledge and gives the teacher valuable information about how students understand the facts of the problem and about the strategies that they may use to solve the problem.

By allowing for this discussion, the teacher has provided the students with an opportunity to stretch their thinking about the problem and has opened up avenues for them to travel towards a solution.

This seemingly simple question often provides details about students' thinking.

Students begin to connect "real life" mathematics with "school" mathematics when they are given the opportunity to talk about personal experiences.

Having students restate the problem, and the information associated with it, provides students with the opportunity to demonstrate their oral communication skills.

Recording what is known about the problem gives students valuable information that they may need

The teacher directs the students: *"Find as many solutions to the problem as possible. Then decide which solution is the most likely and explain why."*

to refer to during the problem-solving process and provides the teacher with an opportunity to model good written communication skills.

The teacher says, *"You may work with a partner or by yourself,"* and then reminds the students that each one will be responsible for recording and sharing what he or she has discovered.

#### *Working in Pairs/Groups*

Although some students choose to work with a partner, each individual student is responsible for recording his or her own thinking and solutions. The teacher can use these recorded work samples when questioning students individually, to ensure conceptual understanding and effective communication. By using a journal or log as a recording device throughout the year, teachers can refer to anecdotal notes that show growth in problem solving, communication, and understanding.

The teacher reminds students that they may use any of the materials in the classroom to help them.

Some students may approach this problem by using pictures and/or numbers and not use manipulatives at all. In other words, for some students using a manipulative may not be the only way to model the problem and a solution. Such students may find that a pictorial or numbered representation is an effective way to model the problem and to communicate their thinking.

The teacher clarifies that the students know what to do if they have difficulty getting started. Student suggestions might include:

- Talk to your partner or ask a classmate.
- Look at the problem information recorded earlier by the teacher, or look at problem-solving charts displayed in the classroom.
- Ask the teacher for help if he or she is not busy with other students.

If students are aware of various steps to take when they are having difficulty getting started, the number of times that they will approach the teacher for help will be reduced. These suggestions encourage students to use their reading, comprehension, speaking, and listening skills with their peers before approaching the teacher.

Before starting, the students are reminded of what the teacher's role will be. *"I will be coming around the room as you work. I may ask some questions and take some notes."*

### **Working on It** (Make a plan, Carry out the plan) *(Facilitating learning)*

The teacher facilitates learning by:

- providing situations in which students are trying their own strategies;
- offering guidance and redirection through questioning;
- giving assistance to those who require it and allowing the others to solve the problem independently.

In this *independent* and *shared experience*, the teacher encourages students to consider the problem information that they have discussed and to choose an appropriate strategy that will assist them in representing

the situation. Manipulative materials are necessary for some students and continue to be a good choice for all students. By encouraging students to make their own choices, the teacher affords all students the opportunity to choose and test their own strategies, to construct their own understanding of the problem and its solutions, and to communicate that understanding to others.

The teacher visits a pair of students who are working with the Grizzly Bear cabin of 4 campers and 3 pizzas.

The teacher asks, *"What strategy did you use for the campers in the Grizzly Bear cabin?"*

Jaspreet replies, *"We decided to cut 2 pizzas into halves and give each camper half, and then cut the remaining pizza into fourths and give each camper another fourth of a pizza."*

*"Why did you choose to divide the pizza into halves and fourths?"*

Evan responds, *"We started with halves because they made the biggest pieces, but then we were left with 2 halves for 4 campers so we had to use fourths."*

The teacher ponders, *"What would happen if you cut the pizzas into halves and fourths for the other cabins?"*

The next pair of students used a similar strategy for the Grizzly Bear cabin but changed their strategy for the Snowy Owl cabin, which earned 4 pizzas for 5 campers.

*"Jamaal, why are you and your partner using a different strategy from the one you used for the Grizzly Bear cabin?"*

Jamaal answers, *"Well, we started dividing the pizzas into fourths like we did the first time, but we found out that 4 pieces of pizza don't divide evenly into 5 campers, so we decided to divide the pizzas into fifths."*

*"So how much pizza did each camper get?"*

Jamaal continues referring to the diagram that he and his partner have drawn. *"Each camper would get  $\frac{1}{5}$  from each pizza, so that means in total each one would get  $\frac{4}{5}$  of a pizza."*

As students set to work, the teacher circulates around the room, monitors the discoveries or misconceptions of the students, and guides students with thoughtful questions or prompts.

The teacher uses effective oral questioning to determine the students' understanding of the strategy they are using. A "what would happen if" question helps focus students on the next stage of the investigation.

Sometimes teachers may direct a question to a specific member of the group who may be reluctant to contribute in large-group discussions. The teacher tries to make this a comfortable situation for the student by:

- referring to the strategy as the group's strategy, so that the focus is not on the individual who is speaking;
- calling on the student in a small-group setting to reduce the anxiety that a large audience of listeners might cause.

The role of the teacher is not to solve the problem but to guide, support, and steer the students towards their own discoveries. Through the use of effective questioning and prompts (e.g., those given on pp. 81–84), the teacher guides the students and provides them with opportunities to communicate their strategies and understanding and to reflect on their own thinking.

*"So 5 campers share 4 pizzas, and each camper ends up getting  $\frac{4}{5}$  of a pizza. That's interesting," concludes the teacher.*

The teacher continues to circulate, visiting all the student pairs as they continue to work on the problem.

After the students have worked for an appropriate amount of time, the teacher lets them know that they have five minutes left to work on the problem. She reminds them to leave their work so that others can see it. Five minutes later, the teacher stops all work, saying, *"I would like you to join me at the group area, but on your way, please take a math walk around the room. Some of the solutions are too big to bring to our sharing circle, but I want you to see all of the exciting math that has happened here today!"*

As the teacher circulates and takes note of the different strategies being used, she makes some decisions about how to structure the discussion during the "Reflecting and Connecting" segment of the lesson.

The teacher celebrates and acknowledges the mathematics that students are doing. A teacher's enthusiasm is contagious! The valuing of the work of all the pairs and individuals will encourage all the students to communicate their solutions during the "Reflecting and Connecting" phase of the lesson.

### **Reflecting and Connecting** (Look back) *(Reflecting on, extending, and consolidating learning)*

In this very important part of the experience, the teacher leads a discussion in which students share their strategies, consider their solutions to the problem, and determine which make the most sense. This discussion validates the various strategies used and consolidates learning for students. Enough time is allotted to allow for the sharing of several examples. (There is not enough time to have all the students present their solutions.) By strategically selecting the partner groups who will present in a particular order, according to the strategies they used, the teacher is structuring the conversation and communication that will result. For many students, the discussion, questioning, and sharing that occur during this phase allow them to make connections with their own thinking and to internalize a deeper understanding of the mathematical concepts.

The teacher asks a student pair to share their discoveries. *"Evan, could you and your partner come up and share your solution?"*

*"We found an answer for the Grizzly Bear cabin by dividing the pizza into halves and fourths. But then we got stuck because we couldn't get the pizzas to divide evenly for the Snowy Owl cabin."*

The teacher attempts to clarify Evan and his partner's thinking for the class, as well as for the two of them, by asking, *"How much pizza does each camper from the Grizzly Bear cabin receive?"*

*"We found that each camper receives half of a pizza and an extra fourth of a pizza."*

The teacher has purposely chosen as the first to share a pair that have been struggling. The two students have experienced success with part of the problem but are having trouble with the rest of it.

Beginning with students who are experiencing difficulty is only one way to structure the "Reflecting and Connecting" part of the lesson. Teachers should order this segment of the lesson in a variety of ways, depending on the purpose of the lesson, and may want to avoid the perception that the "weak" solutions always occur at the beginning of the reflection process.

Through careful use of language, tone, and gestures the teacher communicates to the class that she values the mathematical thinking and efforts of



The teacher continues to prompt Evan: *"How much does each camper get altogether if you put both of those pieces together?"*

*"Altogether that is  $\frac{3}{4}$  of a pizza."*

The teacher then acknowledges that partner group's contribution to the discussion and calls on another group to continue extending the learning for the class: *"Thank you, Evan, for finding the solution for the Grizzly Bear cabin."*

Before moving on to discuss another cabin solution, the teacher gives the class a reminder: *"Remember that it is important not only to record the strategy you used to find the solution for each cabin but also to explain why you chose this strategy. We have several more cabin solutions to explore before we can answer the question of whether the pizza distribution to each cabin was fair. Your solution to the whole problem must include this type of explanation for all the cabins."*

Evan and his partners show their understanding of what the teacher expects of them in their work by adding, *"Well, we started with the Grizzly Bear cabin and divided the first 2 pizzas into halves because they made the biggest pieces, and then each person would get  $\frac{1}{2}$ . That left 1 pizza, which we divided into fourths, so each person would get  $\frac{1}{4}$ . Then we couldn't figure out how to divide the pizzas for the Snowy Owl cabin. We tried using halves and fourths, but they wouldn't divide evenly with the campers."*

*"Is there anyone else who used a similar strategy and found a way to apply it for the other cabins?"*

Hands go up.

*"Jamaal, would you and your partner come up and share your strategy?"*

*"We used the same strategy as Evan's to find the answer for the Grizzly Bear cabin. We had the same problem when we got to the Snowy Owl cabin, but then we tried dividing the pizza into fifths instead of fourths and it worked."*

*"Why did you use fifths?"*

all students in the problem-solving process, regardless of whether they have arrived at a solution for all parts of the problem.

#### *Clear Expectations*

Students are reminded that they are expected to share their solution, explain why they chose it, and explain the process and procedures they used in arriving at it. Junior students should begin to defend their mathematical thinking effectively and should be able to justify their reasoning.

When students are working with a strategy that is making sense to them, they need to explore it, even if they may not always be successful in coming to a solution. These moments of exploring a strategy that is only partially successful are significant learning opportunities.

#### *Problem Solving and Communication Strategies*

It is important to continue to accept and encourage a variety of problem-solving and communication strategies. Students should, however, be able to explain why they chose a particular strategy and to reflect on how effective they believe that strategy was in helping them find an efficient solution to the problem. The explanation could be oral or written and could involve the use of drawings, diagrams, and/or manipulatives.

*"We remembered that we used fourths for dividing pizza for 4 people, and we thought fifths would work for 5 people. So we started handing out fifths of a pizza to all the Snowy Owl campers until all the pizza was gone."*

The teacher then guides the discussion of the other cabin solutions by asking probing questions. She begins, *"Does anyone know a way to describe the strategy these students used so that it will work for all the cabins?"*

Jaspreet responds, *"I think I know. You divide the pizzas into the same number of pieces as campers. In the Grizzly Bear cabin, you divide the pizza into fourths because there are 4 campers, but in the Snowy Owl cabin you can divide the pizzas into fifths because there are 5 campers."*

*"I wonder if that strategy will work for all the cabins?"* the teacher asks, and goes on to say, *"Tomorrow we will continue to explore the amount of pizza received by each camper in the other cabins and determine whether the pizza distribution was fair. Tonight for homework look over your solutions and the strategies you and your partner used, and think about the strategies that we discovered worked for the Grizzly Bear cabin and the Snowy Owl cabin. Will either of those strategies work for the Caribou and the Salmon cabins?"*

#### *Teacher Questioning*

Here the teacher is asking questions to guide the discussion, to emphasize the mathematics, and to build connections between solutions and concepts in order to deepen understanding for all students.

As students respond to the questioning, the teacher has the opportunity to find out what students know, where they will need further instruction, and what experiences to plan for next.

The teacher concludes by pointing forward to the next day's lesson, in which the idea of equivalent fractions will be introduced.

In this lesson the teacher models the importance of cooperative learning. The teacher values the opinions and ideas of the students and gives students opportunities to share with one another in a caring environment. The teacher encourages mathematical thinking and the use of mathematical language.

## Oral Communication

Students who talk and listen to one another converse about mathematical concepts gain experience in reflecting, in reasoning, and in developing mathematical language. *Student talk* is one of the most effective ways for students to demonstrate and to improve their understanding. However, listening to the responses of others and articulating understanding are skills that take practice. Students need many opportunities in the classroom to help develop these skills. Oral communication includes talking, listening, questioning, explaining, defining, discussing, describing, justifying, and defending. When students participate in these actions in an active, focused, and purposeful way, they are furthering their understanding of mathematics.

In the early grades, students often have beginning writing skills that do not allow them to demonstrate the full range of their mathematical knowledge. For this reason, oral discussion is especially important in helping the teacher identify both understandings

**1-3**



and misconceptions. Some students will continue to have difficulty with written communication as they move into the higher grades and will continue to need an emphasis on oral forms of communication. From the early grades on, such oral communication is best developed through a focus on explanations based on concrete or pictorial representations. For example, an intuitive teacher will recognize that very sophisticated problem-solving communication is occurring when a student represents solving the problem of how many hands are in a classroom by drawing circles for people and two sticks to represent each person's hands. That teacher will then help the student to articulate in words the mathematical ideas expressed in the drawing.

If students have been given many opportunities for oral communication in the primary grades, they will be skilled in mathematics discussions and oral forms of communication by the time they enter the junior grades. In the junior grades they need equal opportunities to communicate in both oral and written forms. In particular, they need consistent and continual opportunities to explain their thinking and to consolidate their understanding through oral communication before writing.

The use of correct mathematical terminology becomes crucial as students progress through the junior grades. Students must reach agreement about the meanings of words and have commonly shared definitions in order to work through more complex problems as well as to explain and defend their ideas, answers, and strategies.

## PROMOTING ORAL COMMUNICATION

Teachers can promote students' communication skills by providing models of good communication. For example, they can help students develop skills in problem solving by demonstrating the thinking process for solving a problem and by modelling this process in oral dialogue. The teacher might say: "When I try to find the area of the shape, I look to see what information I have. I have the measurements for several of the sides of this shape. I see that I can divide the shape into two rectangles. I can find the area of each rectangle and then add the areas together." Teachers can also have student volunteers explicitly demonstrate what good communication would sound like. For example, a student could orally describe the steps that he or she took when mentally solving the multidigit addition problem  $431 + 598$ . The student might say: "I decided to add the hundreds together and got 900 ( $400 + 500$ ), then I added the tens together and got 120 ( $30 + 90$ ), and then I added the ones together and got 9 ( $1 + 8$ ). So altogether I had  $900 + 120 + 9$ . Then I added  $900 + 120$  and got 1020, and  $1020 + 9$  equals 1029." This attention to the steps involved emphasizes the importance of student metacognition in the learning process.

As students progress through the primary and junior grades and are given consistent practice in and engagement with "thinking about their thinking", they acquire a

### 4-6

**It is important that teachers respect students' cultural norms related to communication. For example, teachers need to be sensitive to new Canadians who come from a culture in which it is inappropriate for students to ask teachers directly for help, to make eye contact when speaking with others, or to express their opinions.**

deeper understanding of problem-solving processes. When students share their thinking processes, the teacher gains deeper access to what they understand and valuable information that can be used to inform instructional decisions.

In addition, the teacher's use of prompts and questions can help students expand their oral responses. For example, in discussing a growth pattern, the teacher could prompt students to include explanations of how they knew that a certain number of tiles would be needed for a particular stage in the pattern, or could ask them to describe the patterns observed in the T-chart.

Teaching students how to use prompts and questions when they work in a group also helps to improve the communication that occurs when students are working in small groups or in pairs. For example, a simple question that students can learn to ask in response to someone's solution to a problem is "How do you know?"

Encouraging students to use "think time" before they begin to communicate ensures that all students have a chance to think about what they are going to say before they say it. The teacher should wait for several students to put up their hands before asking for a response. If it appears that only the students with their hands up first will be asked to speak to the class, the other students will not feel any individual accountability for contributing to a class discussion.

Often students' communication is improved if they have already made a concrete representation of the problem to be discussed and can then use the representation as the focus of the discussion.

## STRATEGIES FOR PROMOTING ORAL COMMUNICATION

There are many specific strategies for promoting oral communication in a mathematical context. Some lend themselves to use with primary students; others are more appropriate for junior learners. To engage the interest of students, teachers should include a variety of strategies.

The following headings represent strategies that can be used for the purpose of fostering oral communication. A description is given for each strategy.

### Think-Pair-Share

**1-3**

**4-6**

- Students are presented with a task (e.g., a problem to solve, a prompt to respond to).
- Each student spends a couple of minutes thinking about the task.
- The students pair up and share their ideas.
- This activity can be extended to think-pair-share-square, in which two pairs of students join together to discuss their solutions further in a group of four.

## Show and Tell

**1-3****4-6**

- Students are presented with a problem to solve.
- The teacher discusses the problem and possible strategies to ensure that students are clear on what is expected.
- Students do some telling (explain the task) to one another to further ensure that everyone is clear.
- Students build representations (concrete or pictorial) and during this experience, they coach one another and show their ideas to others nearby.
- They further show their work through pictures, words, or graphs.
- Everyone gathers together, and students both show and tell their thinking.

(Dacey & Eston, 2002)

## Math Reader's Theatre

**1-3****4-6**

Working in groups or as a whole class, students prepare a script or a storyboard or use a ready-made script (perhaps taken from the Internet) to demonstrate and explain a mathematical concept. For instance, the students can do a dramatic-reading presentation about the concept of number or of quantity or about the steps for solving a problem. Students may also present word problems in reader's theatre format. This approach can help students understand word problems, with which many students struggle.

## Math Forum

**1-3****4-6**

Students take turns preparing problems to bring to the math forum, where they showcase their work and explain their thinking to a small group of students or to the whole class.

## Cooperative Problem Solving

**1-3****4-6**

- Students work in groups of two or four.
- Each student is given a clue or a piece of information related to a common problem.
- The students share their clues with the group.
- The group work together on the problem, using the clues. For example, for a problem of "find my number":
  - Clue for student one: My number is even.
  - Clue for student two: My number has two digits.
  - Clue for student three: My number is less than 15.
  - Clue for student four: My number is more than 12.
- Students discuss the solution and come to a consensus. They may choose to record their solution to share with the class.

## Catch the Mistake and Make It Right

1-3

Clements and Callahan (1983) describe the following activity:

The students are introduced to a puppet, Mr. Mixup, who constantly makes mistakes. It is up to the students to catch the mistakes. Mr. Mixup makes mistakes that include (a) forgetting to count an object, (b) counting an object more than once, (c) making errors in a counting sequence, or (d) reporting the cardinal value of the set as a number different from the last one spoken. When the students catch a mistake, the teacher asks, "What happened?" and "Can you explain to Mr. Mixup why it's a mistake and tell him how he can be more careful the next time he counts?"

## Prove It or Disprove It

4-6

In classes with older students, the teacher makes conjectures that are either true or false – for example, "An odd number added to an odd number always equals an even number" or "If you add two numbers, you always get a bigger number." The students then, through discussion, prove or disprove the statement. The conjecture may be modified, if necessary, to make it true – for example, "When you add two numbers, neither of which is zero, you get a bigger number than either of the numbers you added."

## Written Communication

Writing in mathematics is a valuable learning and assessment tool because it provides a unique glimpse into students' mathematical understanding. Teachers can use the information gained to tailor their instructional strategies to match the needs of students. Writing also helps students think about and articulate what they know. Having to commit something to paper requires reflection about the mathematics itself, compelling students to justify their logic in their explanations of a solution. Finally, writing shows students' thinking to their teachers and to parents, and provides evidence for assessment.

**Students need time to talk and to listen before they begin to write in the mathematics classroom.**

Before beginning any writing task, students need opportunities to express their ideas orally and to listen to the ideas of others. The quality of the written product is significantly improved by the opportunity to participate in a class dialogue before writing. Oral communication in the mathematics classroom is very public. Ideas "pop out" without editing or revision. Meaning is negotiated or elaborated on by the class as a whole. This opening up of a range of thought and expression is why student talk that occurs before students begin writing is such an important component of written communication for students in all grades.

*"It is this 'rough-draft' talk that allows peers and teachers a window into each other's thinking. As we talk with freshly fashioned ideas in our minds, we all witness the birth of still further ideas."*

(Whitin & Whitin, 2000, p. 2)

Understanding and using the appropriate language of mathematics is crucial to students' learning of mathematics and to their ability to communicate effectively in writing about their mathematics learning. When teachers model and expect correct mathematical language, students quickly replace terms like "plussing" and "take away" with "addition" and "subtraction". Teachers can also provide students with helpful language cues by having visual references in the classroom that include words, pictures, and symbols. (See p. 6.26 and see the subsection "Word Walls" in Chapter 7: Classroom Resources and Management, in Volume Three, for more information on math word walls.) As students develop computer skills, they can use mathematics websites to explore and find definitions of mathematical terms, pictures of shapes, geometric patterns, pictures of nets and skeletons, and so forth. All of these help students understand the language of mathematics and develop the ability to use it in their writing.

In Junior and Senior Kindergarten and Grades 1–3, students will begin by drawing pictures or diagrams to communicate their thinking; eventually, as their writing skills develop, they will add words. Crucial for students in this process – as has already been stated – is the opportunity to talk about their writing before they begin.

**K 1-3**

In Grades 4–6, most students will have more developed writing and recording skills that allow them to communicate their understanding in a variety of ways. After teachers have modelled different ways of recording mathematical thinking and solutions, and have given students multiple opportunities to practise those ways, students in the junior grades will be able to incorporate into their answers words, symbols, numbers, diagrams, charts, and organizers such as webs. Teachers should try to provide students with opportunities to use modes of communication other than words to show what they know, including graphic organizers (e.g., tables) and technology (e.g., computer-generated graphs). However, it is still important that junior students have an opportunity to talk about the math – whether with a partner, in a group, or with the whole class – before being asked to write about it. Talking is like rough-draft thinking and should always precede written tasks.

**4-6**

The amount of writing that can be expected of students at different grade levels will vary considerably. In Kindergarten and Grade 1, pictures with little or no written text, but with an oral description, are appropriate. Written communication can be modelled for young students by a classroom volunteer or teacher who scribes for the student. As students move through the later months of Grade 1 and into Grades 2 and 3, the expectation for writing (as well as for oral communication) should increase. As students progress through the junior grades, the quality and the quantity of their written responses should increase significantly. The assessment or the evaluation of written responses should focus on students' ability to communicate their understanding of the mathematical concepts, rather than on the neatness, spelling, or grammar of the responses.

**K 1-3**

**4-6**

## PROMOTING WRITTEN COMMUNICATION

Teachers can encourage the writing process by modelling it – for example, by providing for shared writing opportunities in mathematics. Such writing opportunities include recording in a class journal, developing written responses together in small peer groupings or together as the whole class, or cooperatively writing out a strategy. Teachers can also support the development of the writing process by providing writing organizers, such as brainstorming activities, word webs, concept maps, Venn diagrams, sentence starters, and writing prompts. Teacher questions and prompts help students to better focus the communication on their mathematical ideas and reasoning. Also, where necessary, teachers can scribe for students to help make the link between their drawings or concrete representations and the written word.

In responding to students' written communications, teachers should:

- probe students' thinking and clarify concepts when misconceptions are evident;
- provide feedback;
- provide time for revisions of work;
- comment on interesting ideas;
- ask for further clarification where necessary;
- ask, "How do you know?";
- compare several student examples, asking questions such as "What can we do to make this clearer or more precise?", and work together with the class to revise the samples for clarity or precision.

Mathematical writing needs to be authentic writing that enhances students' representations and processes, not superficial writing that lacks mathematical purpose.

For example, overuse of the instruction "use pictures, words, and numbers" in presenting tasks to students may stifle their own strategies and creativity and lead them to believe that communication is simply a matter of following a checklist: putting information first in pictures, then duplicating it in words, and finally representing it in numbers. The real goal of communication is for students to think about which way is the best or most effective way for them to show their reasoning and then to use that way flexibly to make their reasoning known to others.

## STRATEGIES FOR PROMOTING WRITTEN COMMUNICATION

It cannot be emphasized enough that the most important component of writing is allowing students to talk about their learning before they begin to write. Also, as was stated earlier, some students will continue to have difficulty with written communication as they move into the higher grades and will continue to need this emphasis on oral communication.

There are many specific strategies for promoting written communication in a mathematical context. Including a variety of these strategies in instruction helps students begin to develop the important skill of using writing to explain mathematical concepts. The following headings represent strategies that can be used for the purpose of fostering written communication. A description is given for each strategy.

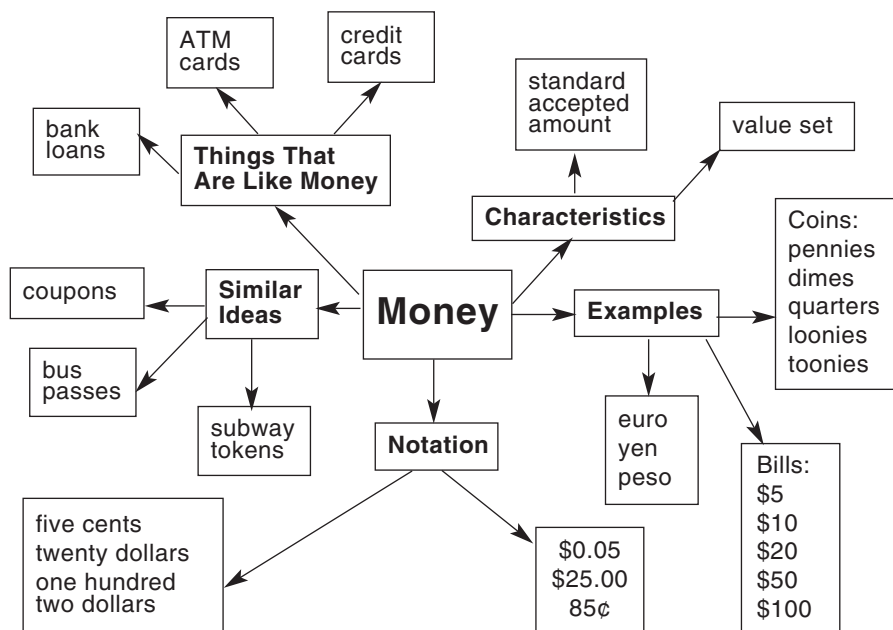
### Mind Mapping

- The teacher leads a class discussion about a concept (e.g., "What do we already know about money?").
- As students suggest ideas, the teacher writes them on the board.
- The teacher makes a list of key words that students used in the discussion.
- The teacher records ideas, using students' terminology.
- As students brainstorm, the teacher draws on the board a mind map showing how one idea is connected with another.

Overuse of the instruction "use pictures, words, and numbers" in presenting tasks to students may stifle their own strategies and creativity and lead them to believe that communication is simply a matter of following a checklist: putting information first in pictures, then duplicating it in words, and finally representing it in numbers.

1-3

4-6





- Students in the later primary grades and in the junior grades can work collaboratively in small groups to create mind maps (e.g., "What do we know about the number 12?").
- Students can also create individual mind maps. Before a concept is introduced, each student can create a mind map (e.g., "What do I know about fractions?"). This will provide the teacher with a glimpse into the student's prior knowledge related to the concept and will allow the teacher to tailor his or her instructional choices to the student's needs. Students can revisit their mind maps and add new learning and understandings in a different colour.

## Model Writing

**1-3**

**4-6**

Students need to see that writing is a useful and natural way to record and communicate mathematical ideas. In the following strategy, the teacher leads the writing process.

- Students contribute thoughts during a class discussion, and the teacher records them on the board, the overhead projector, or chart paper, using bulleted points.
- The teacher then steps back and reflects aloud about the ideas that have been contributed.
- With input from the class, the teacher writes a brief summary of the ideas in paragraph form.
- If appropriate, the teacher adds a drawing, a table, or number sentences to illustrate the idea.
- The teacher talks about what has been written from a critical perspective, to make sure that the paragraph conveys the ideas intended.
- Finally, the teacher makes revisions to demonstrate that the ideas as they are first put in writing are not always in the best form. Revisions should focus on clarifying the students' mathematical reasoning, not on writing skills, though the teacher may silently correct errors in grammar or spelling.

## Shared Student Writing

**1-3**

**4-6**

By sharing some of the writing of their peers, students see models of good mathematical writing.

- The teacher selects a good piece of mathematical writing (e.g., a journal entry, a reflection, a problem, or a story) done by a student and, with the student's permission, displays it on the overhead projector or copies it onto a chart.
- The students discuss the positive characteristics of the writing sample, emphasizing the mathematical content, not the writing style or conventions.
- The teacher highlights the mathematical reasoning and ideas, and furthers the students' understanding of the mathematical concepts in the piece by asking probing questions.



## Group Solution Writing

**1-3****4-6**

- As students work together in small groups to solve a problem, they record their solution process on chart paper or on an overhead transparency.
- During the work session, the teacher circulates and observes the different strategies being used by the groups.
- The teacher selects a few of the students' writing samples for presentation to the class. The reason for choosing the particular group solutions might be to highlight certain problem-solving strategies or to focus on particular mathematical discoveries made by the students.
- Student groups present their solutions to the class. The solutions are then discussed, and the teacher highlights the mathematics and the strategies used.

This group solution-writing process allows students to present their thinking in a non-threatening environment. In this process, both oral and written communication in mathematics are fostered.

This strategy is particularly effective with junior learners but can be used with younger students to encourage appropriate mathematical writing. With younger students the teacher could record the problem-solving process for the group, using a combination of words, numbers, and/or pictures, and then have the students present their ideas to the class.

## Think-Talk-Write

**1-3****4-6**

- The teacher poses a problem or a question, or provides a prompt. For example:
  - "How are a cube and a rectangular prism the same?"
  - "What is division?"
  - "How do you find the perimeter of a square?"
  - "What patterns can you find in the classroom?"
  - "The steps I followed were . . ."
- Students take time to reflect and collect their thoughts.
- Working in small groups, students take turns talking about their solution or answer.
- Students then write a response to the problem, question, or prompt.

## Thinking Windows

**1-3****4-6**

- The teacher invites one student or several students to explain on an overhead transparency their thinking about a given topic.
- The teacher encourages students to use words, symbols, and/or diagrams, and provides students with coloured overhead markers.

- Students talk about their thinking as they work on the overhead, or after they have completed their transparency.
- Other students then ask questions, request further information, or provide constructive feedback.

## Place Mat

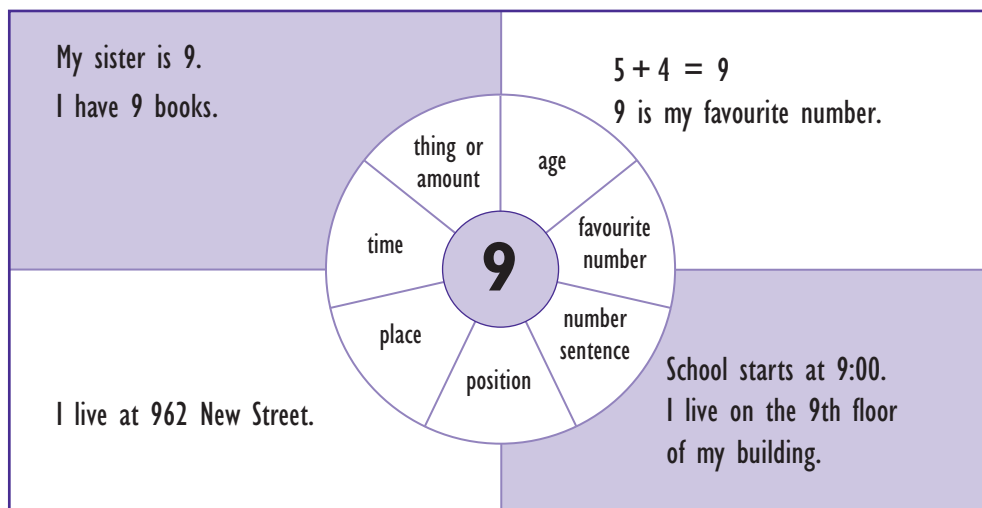
**1-3**

**4-6**

- Students are given a problem, a scenario, or a story to respond to.
- They work in groups of four.
- Each student writes his or her response (pictorial or written) in one quadrant of the place mat (a large sheet divided in four with a centre circle).
- Each student then shares his or her response with the group.
- A joint response, captured in jot notes and/or pictures, is summarized in the circle in the middle of the place mat, and this joint response is shared with the whole class.

(Bennett & Rolheiser, 2001)

### Sample Place Mat: Tell Me All You Know About the Number 9



## Procedural Writing

**1-3**

**4-6**

- Students are given the task of explaining the procedure for finding a solution. For example: "Explain how to multiply  $6 \times 2$  to someone who hasn't learned how."
- The teacher introduces students to features of procedural writing, which include a definite structure (a statement of what is to be done, a list of materials, a list of sequenced steps to be followed), the use of simple imperative statements, and the use of words that indicate sequence (e.g., *first*, *next*, *then*, *after*). For example,

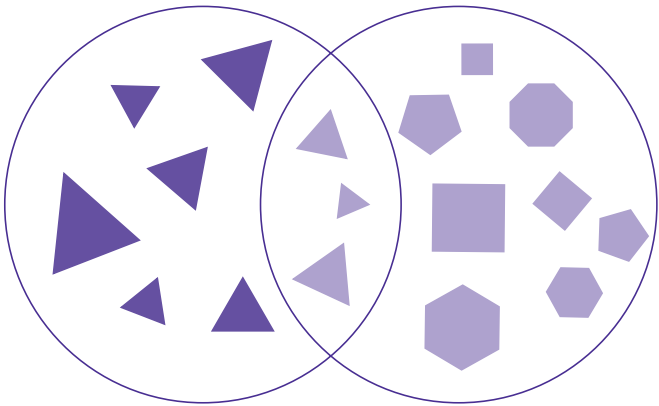
a student might begin the sequence of steps by writing: "First, take two cups and some counters."

- Students write the procedure, give it a title (e.g., "How to multiply  $6 \times 2$ "), and keep it as a writing sample.

**Graphic Organizers**

To convey their understanding of concepts, students arrange information visually, using organizers such as Venn diagrams, flow charts, and T-charts. For example, students might use a Venn diagram to show their understanding of sorting by two attributes, shape and colour:

**1-3** **4-6**



Students could use a T-chart to show their organization and understanding of a pattern.

Input	Output
1	4
2	7
3	10
4	13
5	16
•	•
•	•
•	•

The ability to interpret information from graphic organizers and the ability to use graphics organizers to organize and display information are important skills, because students are exposed to graphic organizers in other subject areas, not just in mathematics. Students' skill in applying these tools and their sophistication in interpreting the information displayed in them increases with experience. As students progress through the grades, they will be given opportunities to use the computer to create and analyse information in various types of organizers (e.g., tables, charts, graphs).

## Math Word Wall

**1-3****4-6**

As students learn new mathematical vocabulary, the teacher adds the words to a display that students can easily see while they are making oral or written responses to mathematics problem solving. Understanding and using the appropriate language of mathematics is crucial to students' learning of mathematics. A math word wall serves as a visual reference for students and reinforces correct mathematical terminology (e.g., students will see and use the word "symmetrical" rather than refer to something as the "mirror image"). It is particularly important that precise mathematical vocabulary be modelled for students in the junior grades and be visually available to them as they engage in more complex justifications and conjectures involving meaning that must be understood by everyone in the class. (See the subsection "Word Walls" in Chapter 7: Classroom Resources and Management, in Volume Three, for more information on word walls.)

## Math Strategy Wall

**1-3****4-6**

As students learn a new mathematics strategy, the teacher puts on the classroom wall a description of the strategy, together with a picture or diagram to illustrate it, if appropriate. Alternatively, the strategy could be collated with other strategies in a big book. (See the subsection "Strategy Walls" in Chapter 7: Classroom Resources and Management, in Volume Three, for more information on strategy walls.)

## Individual and/or Class Journals/Logs

**1-3****4-6**

- Students participate in shared whole-class or small-group writing opportunities to explain and represent (e.g., in pictures, words, and/or numbers) their understanding of a mathematical concept or procedure. The teacher elicits student contributions and responses on a given topic, and records the ideas on chart paper. (Depending on the purpose and scope of the activity and the age of the students, one of the students might do the recording.)
- Each student writes in a math log or journal of his or her own, explaining in pictures, words, numbers, and/or charts his or her understanding of a mathematical concept or procedure.
- As students are able to write more independently, they can do more individual journal entries.

## Math Picture Books

**1-3****4-6**

Working as individuals, in pairs, in groups, or as a whole class, students write and illustrate picture books to explain a concept in mathematics or to tell a mathematical story to younger students. The teacher may want to use published picture books as models to introduce this task.

## Poster Projects

**1-3****4-6**

Working as individuals or in pairs, students investigate a mathematics task, concept, or idea, or a real-life link to mathematics, and illustrate and explain their investigation in a poster format. Real-life links to mathematics are made when students see that the mathematics they are investigating is connected with their everyday lives. For example, students in Kindergarten learning about the number 4 might brainstorm about the number 4 for a poster. Their ideas might include 4 wheels on a minivan, 4 legs on a dog, 4 people in a family. Other examples of poster projects involving real-life links to mathematics include a poster of three-dimensional objects found in the home (e.g., cereal boxes, soup cans); a poster showing proportional heights of students in the class as measured against a door; a poster showing relationships between and among measurement concepts (linear, temporal); a poster showing examples of transformations of geometric figures.

## Students' Problem Posing

**1-3****4-6**

Students write their own mathematical problems about things that are important to them and share the problems with the class. (See the subsection "Problem Posing" in Chapter 5: Problem Solving for more information about problem posing.)

## Math Creative Writing

**1-3****4-6**

Presented with a prompt or a concept, students write creatively. Their writing could focus on their attitudes towards mathematics, or on conceptual or procedural information related to mathematics.

## Questions and Prompts for Promoting Communication

The questions and prompts that teachers use will vary to reflect the purpose of the communication that they want to elicit from students. At different times, teachers ask students to:

- retell;
- predict, invent, or problem solve;
- make connections;
- share their representations of mathematical situations;
- reflect on their work;
- share their feelings, attitudes, or beliefs about mathematics.

The suggestions given in the boxes on pages 81–84 are starting points for promoting communication with students.

*Speaking + listening + reading + writing = understanding in math.*

Questions and prompts may be used in a variety of ways. When students are solving problems, the teacher can use the prompts and questions with individuals or groups of students to help assess whether students understand the mathematics, to determine the next instructional steps, or to provide assessment data that can be recorded on a checklist or anecdotal tracker. After a task, the teacher may use these questions to promote discussion and sharing in a large group, or to provide a stimulus for journal entries developed by the class as a whole or by individual students. Asking students reflective questions (e.g., “How does knowing  $3 \times 4$  help you answer the question  $3 \times 5$ ?”) can be key to students’ metacognition, or their thinking about their own thinking. Reflective questions help students think about and reflect on their own learning and understanding.

Teachers will find that it is helpful to make a copy of the prompts and questions in the following pages and have them readily accessible on a clipboard. Then, as teachers circulate among students who are solving problems, they can use the prompts and questions to focus dialogue and discussion, to clarify students’ developing understanding, and to gain insight into students’ processes and thinking. The prompts can also be posted in the classroom, and students can use them to prompt and question one another.

## QUESTIONS AND PROMPTS TO HELP STUDENTS RETELL

1-3

4-6

Questions to pose

- How did you solve the problem?
- What did you do?
- What strategy did you use?
- What math words did you use or learn?
- What were the steps involved?
- How did your strategy work?
- What did you learn today?
- What do(es) the \_\_\_\_\_ mean to you?

Prompts to use

- *I solved the problem by . . .*
- *The math words I used were . . .*
- *The steps I followed were . . .*
- *My strategy was successful because . . .*
- *Area (or other concept) is . . .*
- *Explain to a young child . . .*
- *Draw a picture to show how you solved that problem.*

These questions and prompts help students to tell, list, choose, recite, name, find, describe, explain, illustrate, and summarize.



## QUESTIONS AND PROMPTS TO HELP STUDENTS PREDICT, INVENT, OR PROBLEM SOLVE

**1-3****4-6**

### Questions to pose

- What would happen if ... ?
- What decisions can you make from the pattern that you discovered?
- How else might you have solved the problem?
- Will it be the same if we use different numbers?
- What things in the classroom have these same shapes?
- How is this pattern like addition?
- What would you measure it with? Why?
- How are adding and multiplying the same?

### Prompts to use

- *Prove that there is only one possible answer to this problem.*
- *Convince me!*
- *Tell me what is the same about ...*

These questions and prompts help students to create, plan, design, predict, imagine, devise, decide, justify, defend, solve, and debate.

## QUESTIONS AND PROMPTS TO HELP STUDENTS MAKE CONNECTIONS

**1-3****4-6**

### Questions to pose

- What does this make you think of?
- What other math can you connect with this?
- When do you use this math at home? At school? In other places?
- Where do you see \_\_\_\_\_ at school? At home? Outside?
- How is this like something you have done before?

### Prompts to use

- *This new math idea is like ...*
- *I thought of ...*
- *I did something like this before when ...*
- *We do this at home when we ...*
- *I remember when we ...*

These questions and prompts help students to connect, relate, refer, imagine, describe, and compare.

**QUESTIONS AND PROMPTS TO HELP STUDENTS  
SHARE THEIR REPRESENTATIONS**

**1-3**

**4-6**

Questions to pose

- How have you shown your thinking (e.g., picture, model, number sentence)?
- Which way (e.g., picture, model, or number sentence) shows what you know best?
- What type of mental strategy (picture, rhyme) did you use?
- How will you remember this next time?
- How have you used math words to describe your experience?
- How did you show it?
- How would you explain \_\_\_\_\_ to a student in Grade \_\_\_\_? (a grade lower than the one the student is in)

Prompts to use

- *I decided to use a . . .*
- *A graph (table, T-chart, picture) shows this the best because . . .*
- *I could make this clearer by using a . . .*
- *A way I can remember this is to . . .*
- *The math words that help someone understand what I did are . . .*

These questions and prompts help students to share, show, describe, demonstrate, and represent.

**QUESTIONS AND PROMPTS TO HELP STUDENTS  
REFLECT ON THEIR WORK**

**1-3**

**4-6**

Questions to pose

- What mathematics were you investigating?
- What questions (feelings) arose as you worked?
- What were you thinking when you made decisions or selected strategies to solve the problem?
- What changes did you have to make to solve the problem?
- What was the most challenging part of the task?
- How do you know?
- How does knowing \_\_\_\_\_ help you to answer the question \_\_\_\_\_?

Prompts to use

- *A question I had was . . .*
- *I was feeling really . . .*
- *When I decided to \_\_\_\_\_, I was thinking . . .*
- *I found \_\_\_\_\_ challenging because . . .*
- *The most important thing I learned in math today (or this week) is . . .*

These questions and prompts help students to analyse, compare, contrast, test, survey, classify, sort, show, use, apply, and model.

## QUESTIONS AND PROMPTS TO HELP STUDENTS SHARE THEIR FEELINGS, ATTITUDES, OR BELIEFS ABOUT MATHEMATICS

1-3

4-6

### Questions to pose

- What else would you like to find out about \_\_\_\_\_?
- How do you feel about mathematics?
- How do you feel about \_\_\_\_\_?
- What does math remind you of?
- How can you describe math?

### Prompts to use

- *The thing I like best about mathematics is . . .*
- *The hardest part of this unit on \_\_\_\_\_ is . . .*
- *I am confused by . . .*
- *I need help with \_\_\_\_\_ because . . .*
- *Write to tell a friend how you feel about what we are doing in mathematics.*
- *Mathematics is like \_\_\_\_\_ because . . .*
- *Today, I felt . . .*
- *I feel good when . . .*

These questions and prompts help students to share, reflect, describe, compare, and tell.



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